

## **EXAMINATION PAPER**

Examination Session: May/June

2020

Year:

Exam Code:

MATH1071-WE01

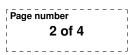
Title:

Linear Algebra I

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks.
	Please start each question on a new page. Please write your CIS username at the top of each page.
	Show your working and explain your reasoning.

Revision:



## SECTION A

**Q1** 1.1 Let  $\Pi$  be the plane in  $\mathbb{R}^3$  passing through the points

$$\begin{pmatrix} 3\\ 3\\ -1 \end{pmatrix}, \quad \begin{pmatrix} 1\\ 3\\ 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 2\\ 1\\ -1 \end{pmatrix}.$$

- (i) Find the equation of  $\Pi$  in the form ax + by + cz = d.
- (ii) If L is the line in  $\mathbb{R}^3$  passing through both the origin and the point  $\begin{pmatrix} 1\\ 2\\ 1 \end{pmatrix}$ , at what point in  $\mathbb{R}^3$  does L intersect  $\Pi$ ?
- **1.2** Calculate the area of the parallelogram in  $\mathbb{R}^2$  with the following vertices:

$$\begin{pmatrix} 1\\1 \end{pmatrix}, \begin{pmatrix} 2\\2 \end{pmatrix}, \begin{pmatrix} 4\\2 \end{pmatrix}, \text{ and } \begin{pmatrix} 5\\3 \end{pmatrix}$$

Q2 2.1 Find the inverse of the following matrix, clearly outlining your method.

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 2 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 2 \end{pmatrix}$$

**2.2** Determine the value of  $t \in \mathbb{R}$  for which the linear system of equations

has a solution. Find the general solution in this case.

- **Q3** Let  $M_n(\mathbb{R})$  be the real vector space consisting of all  $n \times n$  matrices with real entries.
  - **3.1** Suppose  $A_1, A_2, \ldots, A_k$  are k linearly independent matrices in  $M_n(\mathbb{R})$ , and that P and Q are two invertible matrices in  $M_n(\mathbb{R})$ . Show that the k matrices

$$PA_1Q, PA_2Q, \ldots, PA_kQ,$$

are linearly independent.

**3.2** For any fixed matrix B in  $M_n(\mathbb{R})$ , show that the subset

$$W_B = \{A \in M_n(\mathbb{R}) : AB = -BA\}$$

is a vector subspace of  $M_n(\mathbb{R})$ .

**3.3** Assume A and B are two matrices in  $M_n(\mathbb{R})$  such that AB = -BA. Prove that A and B cannot both be invertible if n is odd.

**Q4** Consider the linear map  $S : \mathbb{R}^4 \to \mathbb{R}^4$  defined by

$$S\begin{pmatrix}x\\y\\z\\w\end{pmatrix} = \begin{pmatrix}x-z\\y-w\\z-x\\w-y\end{pmatrix}.$$

- **4.1** Find the matrix A representing S, with respect to the standard basis vectors of  $\mathbb{R}^4$ , and determine the rank and nullity of A.
- **4.2** Find a basis for ker(S) and im(S) and show that

$$\ker(S) \cap \operatorname{im}(S) = \{\mathbf{0}\}.$$

4.3 Hence, or otherwise, prove that

$$\ker(S) \oplus \operatorname{im}(S) = \mathbb{R}^4.$$

**Q5** Let  $\mathbb{R}[x]_n$  be the (n+1)-dimensional vector space consisting of all polynomials whose coefficients are real and whose degree is at most n.

For each k = 0, 1, ..., n define the linear map  $T_k : \mathbb{R}[x]_n \to \mathbb{R}[x]_n$  by

$$T_k(p(x)) = p'(x) + p(1)x^k,$$

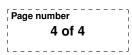
where we write p' for the first derivative of p (with respect to the variable x).

- **5.1** Prove that  $\ker(T_n) = \{0\}$ , where 0 is the zero polynomial. Hence, determine the rank of  $T_n$ .
- **5.2** For any k < n, evaluate  $T_k(p_k(x))$ , where

$$p_k(x) = \frac{k+2}{k+1} - \frac{x^{k+1}}{k+1}.$$

Hence, or otherwise, determine the values of k for which  $T_k$  is an isomorphism.

**5.3** For the case n = 3, write down the matrix representing  $T_2$  with respect to the standard basis of  $\mathbb{R}[x]_3$ .



## SECTION B

Q6 Given the matrix

$$A = \begin{pmatrix} 0 & 2 & -2 \\ 2 & 0 & -2 \\ 0 & 0 & -2 \end{pmatrix} ,$$

find P such that  $P^{-1}AP$  is diagonal, clearly outlining your method. Use this result to compute  $A^{10}$  (brute force calculation is not allowed).

**Q7** Let V be the vector space  $\mathbb{R}[x]_3$  of real polynomials of degree at most three and let  $\mathcal{L}: V \mapsto V$  be the linear operator

$$\mathcal{L}(p(x)) = p(x+1) - \frac{a}{x} \int_0^x p(y) dy$$

with  $p(x) \in \mathbb{R}[x]_3$  and  $a \in \mathbb{R}$ . Find the matrix representing the linear operator  $\mathcal{L}$  on V using the standard basis  $\{1, x, x^2, x^3\}$ . Show that for a = 4 one of the eigenvalues of the operator  $\mathcal{L}$  is equal to -1 and then compute the eigenfunction corresponding to this eigenvalue.

**Q8** 8.1 Find the necessary conditions on the real parameters a and b so that

$$(\mathbf{x}, \mathbf{y}) = 3x_1y_1 + x_1y_2 + ax_2y_1 + 2x_2y_2 + x_3y_2 + x_2y_3 + bx_3y_3$$

defines an inner product on  $V = \mathbb{R}^3$ .

8.2 If  $V = \mathbb{R}^4$  is given the standard inner product, find an orthonormal basis for the subspace determined by the equation

$$x_1 - x_2 + x_3 - x_4 = 0 \,,$$

and then extend this basis to an orthonormal basis for all  $V = \mathbb{R}^4$ .

- **Q9** Let V be a complex vector space with inner product  $\langle , \rangle$  and let  $\mathcal{L} : V \to V$  be a linear hermitian operator with respect to this inner product. First prove that the eigenvalues of  $\mathcal{L}$  must be real and then show that if  $\mathbf{z}, \mathbf{w} \in V$  are eigenvectors of  $\mathcal{L}$  corresponding to different eigenvalues then they must be orthogonal.
- **Q10** Let H be the set of matrices of the form

$$A = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \,,$$

with  $a, b, c \in \mathbb{Z}$ . Show that H is a group with respect to matrix multiplication. Is the group abelian? Justify your answer. [You may assume associativity]