Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2020	MATH1071-WE01

Title:
Linear Algebra I

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. Show your working and explain your reasoning.

Revision:	

ED01/2020

SECTION A

Q1 **1.1** Let Π be the plane in $\mathbb{R}^3$ passing through the points

$$\begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}.$$

(i) Find the equation of Π in the form $ax + by + cz = d$.

(ii) If L is the line in $\mathbb{R}^3$ passing through both the origin and the point $\left(\begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix}\right)$, at what point in $\mathbb{R}^3$ does L intersect Π?

1.2 Calculate the area of the parallelogram in $\mathbb{R}^2$ with the following vertices:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 5 \\ 3 \end{pmatrix}.$$

Q2 **2.1** Find the inverse of the following matrix, clearly outlining your method.

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 2 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 2 \end{pmatrix}$$

2.2 Determine the value of $t \in \mathbb{R}$ for which the linear system of equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= t, \\ 2x_1 + 2x_2 + 2x_3 &= 1, \\ 3x_1 + 2x_2 + x_3 &= 1, \end{aligned}$$

has a solution. Find the general solution in this case.

Q3 Let $M_n(\mathbb{R})$ be the real vector space consisting of all $n \times n$ matrices with real entries.

3.1 Suppose $A_1, A_2, \ldots, A_k$ are k linearly independent matrices in $M_n(\mathbb{R})$, and that P and Q are two invertible matrices in $M_n(\mathbb{R})$. Show that the k matrices

$$PA_1Q, \; PA_2Q, \; \ldots, \; PA_kQ,$$

are linearly independent.

3.2 For any fixed matrix B in $M_n(\mathbb{R})$, show that the subset

$$W_B = \{A \in M_n(\mathbb{R}) \; : \; AB = -BA\}$$

is a vector subspace of $M_n(\mathbb{R})$.

3.3 Assume A and B are two matrices in $M_n(\mathbb{R})$ such that $AB = -BA$. Prove that A and B cannot *both* be invertible if n is odd.

Q4 Consider the linear map $S : \mathbb{R}^4 \to \mathbb{R}^4$ defined by

$$S \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} x - z \\ y - w \\ z - x \\ w - y \end{pmatrix}.$$

4.1 Find the matrix A representing S, with respect to the standard basis vectors of $\mathbb{R}^4$, and determine the rank and nullity of A.

4.2 Find a basis for $\ker(S)$ and $\text{im}(S)$ and show that

$$\ker(S) \cap \text{im}(S) = \{\mathbf{0}\}.$$

4.3 Hence, or otherwise, prove that

$$\ker(S) \oplus \text{im}(S) = \mathbb{R}^4.$$

Q5 Let $\mathbb{R}[x]_n$ be the $(n+1)$-dimensional vector space consisting of all polynomials whose coefficients are real and whose degree is at most n.

For each $k = 0, 1, \ldots, n$ define the linear map $T_k : \mathbb{R}[x]_n \to \mathbb{R}[x]_n$ by

$$T_k(p(x)) = p'(x) + p(1)x^k,$$

where we write p' for the first derivative of p (with respect to the variable x).

5.1 Prove that $\ker(T_n) = \{0\}$, where 0 is the zero polynomial. Hence, determine the rank of T_n.

5.2 For any $k < n$, evaluate $T_k(p_k(x))$, where

$$p_k(x) = \frac{k+2}{k+1} - \frac{x^{k+1}}{k+1}.$$

Hence, or otherwise, determine the values of k for which T_k is an isomorphism.

5.3 For the case $n = 3$, write down the matrix representing T_2 with respect to the standard basis of $\mathbb{R}[x]_3$.

SECTION B

Q6 Given the matrix

$$A = \begin{pmatrix} 0 & 2 & -2 \\ 2 & 0 & -2 \\ 0 & 0 & -2 \end{pmatrix},$$

find P such that $P^{-1}AP$ is diagonal, clearly outlining your method. Use this result to compute A^{10} (brute force calculation is not allowed).

Q7 Let V be the vector space $\mathbb{R}[x]_3$ of real polynomials of degree at most three and let $\mathcal{L} : V \mapsto V$ be the linear operator

$$\mathcal{L}\left(p(x)\right) = p(x+1) - \frac{a}{x}\int_0^x p(y)dy\,,$$

with $p(x) \in \mathbb{R}[x]_3$ and $a \in \mathbb{R}$. Find the matrix representing the linear operator $\mathcal{L}$ on V using the standard basis $\{1, x, x^2, x^3\}$. Show that for $a = 4$ one of the eigenvalues of the operator $\mathcal{L}$ is equal to -1 and then compute the eigenfunction corresponding to this eigenvalue.

Q8 **8.1** Find the necessary conditions on the real parameters a and b so that

$$(\mathbf{x}, \mathbf{y}) = 3x_1y_1 + x_1y_2 + ax_2y_1 + 2x_2y_2 + x_3y_2 + x_2y_3 + bx_3y_3$$

defines an inner product on $V = \mathbb{R}^3$.

8.2 If $V = \mathbb{R}^4$ is given the standard inner product, find an orthonormal basis for the subspace determined by the equation

$$x_1 - x_2 + x_3 - x_4 = 0\,,$$

and then extend this basis to an orthonormal basis for all $V = \mathbb{R}^4$.

Q9 Let V be a complex vector space with inner product $< \, , \, >$ and let $\mathcal{L} : V \to V$ be a linear hermitian operator with respect to this inner product. First prove that the eigenvalues of $\mathcal{L}$ must be real and then show that if $\mathbf{z}, \mathbf{w} \in V$ are eigenvectors of $\mathcal{L}$ corresponding to different eigenvalues then they must be orthogonal.

Q10 Let H be the set of matrices of the form

$$A = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix},$$

with $a, b, c \in \mathbb{Z}$. Show that H is a group with respect to matrix multiplication. Is the group abelian? Justify your answer. [You may assume associativity]