Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2020	MATH1551-WE01

Title: Maths For Engineers and Scientists

Time (for guidance only):	3 hours	
Additional Material provided:	Formula sheet	
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. Show your working and explain your reasoning.

Revision:	

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Q1 **1.1** Let $z = 1 - i\sqrt{3}$. For which integers $m \in \mathbb{Z}$ is z^m a real number ?
Find the modulus and argument of every possible value of z^i.

1.2 State de Moivre's Theorem and use it to express $\tan(4\theta)$ as a function of $\tan\theta$.

1.3 Find all complex numbers z satisfying the equation $\sin(2z) = 3$.

Q2 **2.1** Compute the limit of the sequence $s_n = \sqrt{n^2 + 4n} - \sqrt{n^2 + n}$ as $n \to \infty$.

2.2 Compute the following limits, stating any standard results you use.

$$\lim_{x\to 0} \frac{\sqrt{1+x^2}-1}{x^2}, \qquad \lim_{x\to 1} \frac{\sin(\pi x - \pi)}{1-x}, \qquad \lim_{x\to\infty} 2^{-x}\cos(\sin x).$$

2.3 State the definitions of continuity and differentiability of a function $f(x)$ at a point $x = a$ in terms of limits. Suppose that a function $f(x)$ is continuous but *not* differentiable at $x = 0$. Prove that $g(x) = xf(x)$ is differentiable at $x = 0$ and find its derivative at that point.

Q3 **3.1** State Leibniz' Rule for finding the nth derivative of a product $f(x) = u(x)v(x)$. Apply it to find the third derivative of $f(x) = (\ln x)\sin(2x)$.

3.2 Find the degree 2 Taylor polynomial $p(x)$ for the function $f(x) = x^{2/3}$ about the point $x = 1$. Use the bound on the remainder term to show that

$$|f(x) - p(x)| \leq \frac{4}{81000} \qquad \text{for all } 1 \leq x \leq 1.1.$$

3.3 Write out a Newton-Raphson iteration scheme that will give a sequence of better and better approximations to $2^{1/5}$. If an initial guess is $2^{1/5} \approx 1$, find the next approximation, expressing your answer as a fraction.

Q4 **4.1** Find a Cartesian equation for the plane in $\mathbb{R}^3$ passing through the three points $A = (1, 2, 3)$, $B = (3, 4, 4)$ and $C = (-1, 3, 6)$.

4.2 Find a parametric equation for the line of intersection of the two planes

$$x + 2y + 4z = 3 \qquad \text{and} \qquad x - y + z = 3.$$

What is the cosine of the angle θ between the two planes ?

4.3 Given a function $f(x, y)$ and the change of variables $u = x + y$, $v = x - y$, use the chain rule to express $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial^2 f}{\partial x \partial y}$ in terms of partial derivatives with respect to u and v.

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Q5 **5.1** Determine all the critical points of the function

$$f(x, y) = x^3 - 3x^2 - \cos(\pi y)$$

and classify each as a local minimum, local maximum or saddle point.

5.2 For which value of the constant a is the following differential equation

$$\left(x^2 + 2xy + 4y^3\right)\frac{dy}{dx} + \left(3x^2 + axy + y^2\right) = 0$$

exact ? When a takes this value, find the general solution to the equation. (You may leave your answer in implicit form.)

Q6 Find the general solution to the differential equation

$$y'' - 4y' + 4y = 4x + 4\sin(2x)$$

and hence find the solution satisfying $y(0) = y'(0) = 1$.

Q7 **7.1** Use Gaussian elimination to find the inverse of the matrix

$$\begin{pmatrix} 2 & 3 & 2 \\ 1 & 2 & 1 \\ 4 & 3 & 5 \end{pmatrix}.$$

7.2 Rearrange the following system of equations into a form you know will converge under the SOR iterative method for any initial point $\mathbf{x}^{(0)}$, stating why.

$$\begin{aligned} x + 4y + 2z &= 3 \\ 3x - y - z &= 10 \\ -2x - 3y + 6z &= -3 \end{aligned}$$

Write out the resulting SOR iteration with $\omega = 0.5$ and hence calculate $\left(x^{(1)}, y^{(1)}, z^{(1)}\right)$ given that $\left(x^{(0)}, y^{(0)}, z^{(0)}\right) = (2, 2, 0)$.

Q8 Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 3 & -3 \\ 4 & -5 \end{pmatrix}$$

and hence diagonalise A. Use your result to find the solution to the system

$$\begin{aligned} \dot{x} &= 3x - 3y \\ \dot{y} &= 4x - 5y \end{aligned}$$

satisfying the initial conditions $x(0) = 2$, $y(0) = 0$.

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Formula sheet for **Mathematics for Engineers & Scientists (MATH1551/WE01)**

TRIGONOMETRIC FUNCTIONS

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

$$\cos^2 A + \sin^2 A = 1$$

$$1 + \tan^2 A = \sec^2 A$$

$$1 + \cot^2 A = \operatorname{cosec}^2 A$$

$$\sin A\cos B = \frac{1}{2}\left(\sin(A+B) + \sin(A-B)\right)$$

$$\cos A\cos B = \frac{1}{2}\left(\cos(A+B) + \cos(A-B)\right)$$

$$\sin A\sin B = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$$

$$\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

$$\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

$$\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

$$\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

HYPERBOLIC FUNCTIONS

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh^{-1} x = \ln\left(x \pm \sqrt{x^2-1}\right)$$

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$$

$$\cosh(iA) = \cos A$$

$$\sinh(iA) = i\sin A$$

$$\cosh^2 A - \sinh^2 A = 1$$

$$\cosh(A+B) = \cosh A\cosh B + \sinh A\sinh B$$

$$\sinh(A+B) = \sinh A\cosh B + \cosh A\sinh B$$

$$\tanh(A+B) = \frac{\tanh A + \tanh B}{1 + \tanh A\tanh B}$$

ELEMENTARY RULES FOR DIFFERENTIATION AND INTEGRATION

$$(u+v)' = u'+v', \quad (uv)' = u'v+uv', \quad \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}, \quad (u(v))' = u'(v)v', \quad \int u'v\,dx = uv - \int uv'\,dx$$

TAYLOR'S THEOREM

Taylor approximation: $f(x) \approx p_{n,a}(x) = f(a) + f'(a)(x-a) + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n$

and if $|f^{(n+1)}(x)| \le M$ for $c \le x \le b$ then $|f(x) - p_{n,a}(x)| \le \frac{|x-a|^{n+1}}{(n+1)!}M$

DIFFERENTIAL OPERATORS

$$\text{grad } f = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$\text{div } \mathbf{A} = \nabla\cdot\mathbf{A} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot(A_1, A_2, A_3) = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

$$\text{curl } \mathbf{A} = \nabla\times\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} = \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\mathbf{k}$$

TABLE OF DERIVATIVES

$y(x)$	$\dfrac{dy}{dx}$
x^n	nx^{n-1}
$\ln x$	x^{-1}
e^x	e^x
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\tan^{-1} x$	$\dfrac{1}{1+x^2}$
$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$\dfrac{-1}{\sqrt{1-x^2}}$
$\sinh^{-1} x$	$\dfrac{1}{\sqrt{1+x^2}}$
$\cosh^{-1} x$	$\dfrac{1}{\sqrt{x^2-1}}$

TABLE OF INTEGRALS

$f(x)$	$\displaystyle\int f(x)\,dx$
x^n	$\dfrac{x^{n+1}}{n+1} \quad (n \neq -1)$
x^{-1}	$\ln\lvert x\rvert$
e^x	e^x
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\tan x$	$-\ln\lvert\cos x\rvert$
$\operatorname{cosec} x$	$-\ln\lvert\operatorname{cosec} x + \cot x\rvert$
$\sec x$	$\ln\lvert\sec x + \tan x\rvert$
$\cot x$	$\ln\lvert\sin x\rvert$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\ln\lvert\cosh x\rvert$
$\dfrac{1}{a^2+x^2}$	$\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\dfrac{x}{a}\right) \quad (a > x)$
$\dfrac{1}{\sqrt{a^2+x^2}}$	$\sinh^{-1}\left(\dfrac{x}{a}\right)$
$\dfrac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}\left(\dfrac{x}{a}\right) \quad (x > a)$

CRITICAL POINTS

Local maximum: $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial y} = 0$ and $\dfrac{\partial^2 f}{\partial x^2}\dfrac{\partial^2 f}{\partial y^2} - \left(\dfrac{\partial^2 f}{\partial y \partial x}\right)^2 > 0$ and $\dfrac{\partial^2 f}{\partial x^2} < 0$

Local minimum: $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial y} = 0$ and $\dfrac{\partial^2 f}{\partial x^2}\dfrac{\partial^2 f}{\partial y^2} - \left(\dfrac{\partial^2 f}{\partial y \partial x}\right)^2 > 0$ and $\dfrac{\partial^2 f}{\partial x^2} > 0$

Saddle point: $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial y} = 0$ and $\dfrac{\partial^2 f}{\partial x^2}\dfrac{\partial^2 f}{\partial y^2} - \left(\dfrac{\partial^2 f}{\partial y \partial x}\right)^2 < 0$

Inconclusive: $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial y} = 0$ and $\dfrac{\partial^2 f}{\partial x^2}\dfrac{\partial^2 f}{\partial y^2} - \left(\dfrac{\partial^2 f}{\partial y \partial x}\right)^2 = 0$

ITERATION METHODS

$$A\,\mathbf{x} = \mathbf{b}, \qquad A = D - L - U, \qquad T_j = D^{-1}(L+U), \qquad T_g = (D-L)^{-1}U$$

Jacobi's Method:

$$D\,\mathbf{x}^{(k+1)} = \mathbf{b} + (L+U)\,\mathbf{x}^{(k)}, \qquad \mathbf{x}^{(k+1)} = D^{-1}\,\mathbf{b} + D^{-1}(L+U)\,\mathbf{x}^{(k)}$$

Gauss-Seidel Method:

$$D\,\mathbf{x}^{(k+1)} = \mathbf{b} + L\,\mathbf{x}^{(k+1)} + U\,\mathbf{x}^{(k)}, \qquad \mathbf{x}^{(k+1)} = (D-L)^{-1}\,\mathbf{b} + (D-L)^{-1}U\,\mathbf{x}^{(k)}$$

SOR Method:

$$\mathbf{x}^{(k+1)} = (1-\omega)\,\mathbf{x}^{(k)} + \omega D^{-1}\left(\mathbf{b} + L\,\mathbf{x}^{(k+1)} + U\,\mathbf{x}^{(k)}\right)$$

Optimal Value: $\omega = \dfrac{2}{1+\sqrt{1-\rho\,(T_j)^2}}$