Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2020	MATH1571-WE01

Title: Single Mathematics B

Time (for guidance only):	3 hours	
Additional Material provided:	Tables: Normal Distribution	
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. Show your working and explain your reasoning.

Revision:	

ED01/2020

Q1 A particle of constant mass m has position vector

$$\mathbf{r}(t) = (\cos(\omega t), \sin(\omega t), 1)$$

as a function of time t.

1.1 Show that the motion of the particle is planar by finding a normal vector $\mathbf{n}$, and a constant c, such that $\mathbf{n} \cdot \mathbf{r}(t) = c$.

1.2 Find the angular momentum of the particle around the origin, and compute the torque acting on the particle.

1.3 Show that the force acting on the particle is a combination of a central force and a term normal to the plane of motion of the particle.

Q2 **2.1** A particle is free to move along the x-axis in one dimension. The particle's velocity v satisfies the equation

$$\frac{dv}{dt} = -v - v^2.$$

If at $t = 0$ the particle is at the origin $x = 0$ moving with velocity $v = 2$, find the particle's position x and velocity v at time $t = 1$.

2.2 Find the general solution to the equation

$$\frac{dP}{dt} = P - tP^4.$$

Q3 A function $f(x)$ has period 2π and is defined to be $f(x) = 1 + |x|$ on the interval $[-\pi, \pi]$.

3.1 Sketch the function in the range $-4\pi < x < 4\pi$.

3.2 Find the Fourier coefficients a_n, b_n if $f(x)$ is expanded in a Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx).$$

3.3 By evaluating the Fourier series at $x = 0$, find the value of the sum

$$S = \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2}.$$

Q4 **4.1** If $f(x, y)$ is a function of x and y where $x = e^{u+v}$ and $y = e^{u-v}$, show that

$$f_{uv} - f_v = x^2 f_{xx} - y^2 f_{yy}\,,$$

where the subscripts indicate partial differentiation. (You may assume that $f_{xy} = f_{yx}$.)

4.2 A cone (with open top) for holding liquid is made from a fixed area, S, of metal. Find the value of the radius r that maximizes the volume V of liquid it holds. What is the corresponding height of the cone and its volume. (You may use that the area of a cone of height h is $S = \pi r\sqrt{h^2 + r^2}$, while its volume is $V = \pi r^2 h/3$.)

4.3 The height of a hill is given by

$$h(x, y) = x^2 - 4x + y^2 + xy.$$

Consider the level-curve where the surface is at sea-level ($h = 0$). If a ball is released from $(x, y) = (1, 0)$ on this curve, in which direction does it roll? Where does it finally come to rest (assuming it loses energy through friction) and at what height below sea-level?

Q5 **5.1** Determine what values of constant parameters a and b makes the following differential exact:

$$df = a\frac{xz}{x^2 + y^2}dx + b\frac{yz}{x^2 + y^2}dy + \ln\left(x^2 + y^2\right)dz$$

5.2 Defining $\boldsymbol{V} = \left(a\frac{xz}{x^2+y^2},\, b\frac{yz}{x^2+y^2}dy,\, \ln\left(x^2 + y^2\right)\right)$ with the values of a, b determined in question **5.1**, determine:

(i) $\nabla \times \boldsymbol{V}$

(ii) $\nabla \cdot \boldsymbol{V}$

(iii) $\nabla \times (z\boldsymbol{V})$

5.3 Show that any vector field $\boldsymbol{V} = (V_1, V_2, V_3)$ that has a z-component satisfying $\frac{\partial V_3}{\partial z} = 0$ and that has the curl and divergence you determined for $\boldsymbol{V}$ in the previous question, has x and y-components that each satisfy the 2 dimensional Laplace's equation: that is

$$\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial y^2} = 0 = \frac{\partial^2 V_2}{\partial x^2} + \frac{\partial^2 V_2}{\partial y^2}.$$

Q6 **6.1** Evaluate the integral

$$\iint_A (x^2 + y^2)dxdy$$

where

(i) A is the annulus $1 \leq x^2 + y^2 \leq 4$

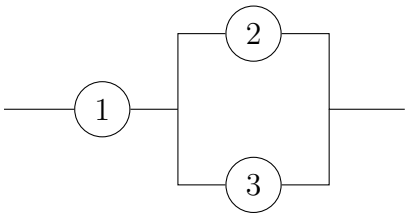
(ii) A is the square $1 \leq x \leq 2$, $1 \leq y \leq 2$

6.2

(i) Show that the function $u + iv = \sinh(x + iy)$ satisfies the Cauchy-Riemann equations, $u_x = v_y$, and $u_y = -v_x$.

(ii) Where is the function $f = \sinh(|x| + i|y|)$ analytic?

Q7 **7.1** Consider the following circuit formed of three components. The circuit works if both component 1 works **and** at least one of component 2 or component 3 works.



Assume that the components work independently and component i works with probability p_i.

(i) Show that the probability that either component 2 or component 3 works is $p_2 + p_3 - p_2 p_3$.

(ii) Using part i. or otherwise, find an expression for the probability that the circuit is working.

(iii) Find the (conditional) probability that the circuit works given that component 2 is working.

(iv) Find the (conditional) probability that component 2 works given that the circuit is working.

7.2 A laboratory tests DNA samples for a particular genetic marker. It is known that a proportion 0.5% of the total population has the marker. The laboratory tests 500 samples a week.

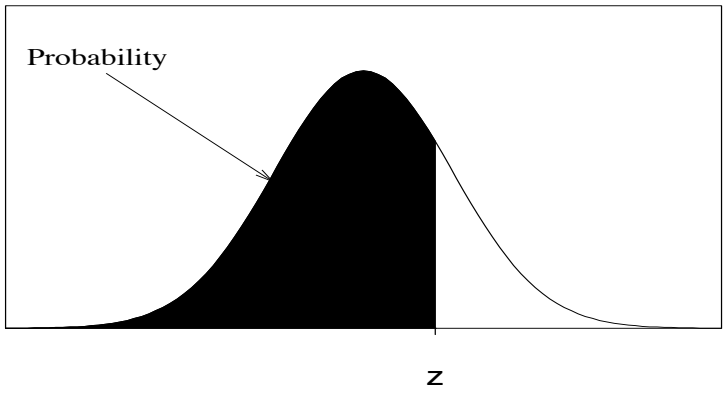
(i) What is the distribution of the number of tests that return positive for the marker in a week? What assumptions did you need to make?

(ii) Using an appropriate approximation, calculate the probability that the number of positive tests is less than or equal to 2.

(iii) Suppose the laboratory keeps a record of the number of positive test results per week for a whole year, i.e., 52 weeks of data. Approximate the probability that the **average** number of positive tests is less than 2.

Probabilities for the standard normal distribution

Table entry for z is the probability lying to the left of z



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998