Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2020	MATH1597-WE01

Title:
Probability I

Time (for guidance only):	2 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. Show your working and explain your reasoning.

Revision:	

ED01/2020

Q1 **1.1** You roll two fair six-sided dice. Find the probability that you get exactly two sixes, given that you get at least one six.

1.2 In a randomly selected group of 100 people, let X be the number of people who were born on Christmas day. Suppose that births are independent, and are equally likely to be on any of the 365 days of the year (ignore leap years).

(i) Write down a numerical expression for $\mathbb{P}(X \geq 1)$, but do not evaluate it.

(ii) Use an appropriate distributional approximation to write down an alternative numerical expression for $\mathbb{P}(X \geq 1)$.

1.3 Suppose that Y is a continuous random variable with probability density function f given by $f(y) = cy(1-y)$ for $0 < y < 1$ and $f(y) = 0$ elsewhere.

(i) Find the value of the constant c.

(ii) Compute $\mathbb{E}(Y)$ and $\mathbb{V}\mathrm{ar}(Y)$.

(iii) Find $\mathbb{E}(1/Y)$.

Q2 Suppose that $X_1, X_2, \ldots, X_{100}$ are independent, identically distributed random variables with $\mathbb{E}(X_i) = 10$ and $\mathbb{V}\mathrm{ar}(X_i) = 4$. Consider

$$q = \mathbb{P}\left(\left| \sum_{i=1}^{100} X_i - 1000 \right| \geq 30 \right).$$

2.1 Use Chebyshev's inequality to give an upper bound on q.

2.2 Use the central limit theorem to give an approximate value for q.

2.3 Give a distribution for the X_i for which your answer to question **2.2** is exact.

In your answer you may use the following table of values of the standard normal cumulative distribution function.

z	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$\Phi(z)$	0.540	0.579	0.618	0.655	0.691	0.726	0.758	0.788	0.816	0.841
z	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
$\Phi(z)$	0.864	0.885	0.903	0.919	0.933	0.945	0.955	0.964	0.971	0.977
z	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3.0
$\Phi(z)$	0.982	0.986	0.989	0.992	0.994	0.995	0.997	0.997	0.998	0.999

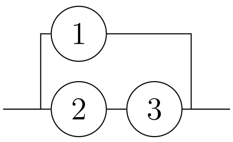
Q3 **3.1** Mice of a certain breed are either grey or white. The colour is determined by a gene which has alleles A (dominant) and a (recessive). Mice of genotype AA or Aa are grey, while mice of type aa are white.

Mortimer mouse is known to be the offspring of parents with genotypes AA and Aa, while Maximilian mouse is the offspring of AA and aa parents. Both are grey males.

(i) If Mortimer were mated to a white female, producing a single offspring, what is the probability that the offspring is grey?

(ii) Melanie mouse is a white female. She is pregnant, and we believe that the father is equally likely to be Mortimer or Maximilian. The pregnancy bears a single offspring, which is grey. Given this information, what is our new probability that Mortimer is the father?

3.2 Consider the following reliability network:



Assume that components fail independently, with $\mathbb{P}(\text{component } i \text{ works}) = p_i$. Find, as a function of p_1, p_2, p_3, the (conditional) probability that component 2 works, given that the system works.

Q4 Denis's picnic hamper contains two batter puddings, two curried eggs, and one kippered herring. Denis draws items one at a time from his hamper, randomly and without replacement, until he obtains a batter pudding. Let X denote the number of kippered herring he draws, and let Y be the number of curried eggs.

4.1 Present in a table the joint probability mass function of X and Y.

4.2 Calculate the marginal probability mass functions of X and Y, and find $\mathbb{E}(X)$ and $\mathbb{E}(Y)$.

4.3 Compute $\mathbb{C}\text{ov}(X, Y)$.

4.4 Find $\mathbb{P}(Y = y \mid X = 0)$ and $\mathbb{P}(Y = y \mid X = 1)$ for all $y \in \{0, 1, 2\}$.

4.5 Verify in this example that $\mathbb{E}(\mathbb{E}(Y \mid X)) = \mathbb{E}(Y)$.

Q5 Recall that $X \sim \text{Po}(\lambda)$ has the Poisson distribution with parameter $\lambda > 0$ if

$$\mathbb{P}(X = x) = \mathrm{e}^{-\lambda}\frac{\lambda^x}{x!}, \text{ for } x = 0, 1, 2, \ldots.$$

5.1 Show that the moment generating function of a random variable $X \sim \text{Po}(\lambda)$ is $M_X(t) = \exp\{\lambda(\mathrm{e}^t - 1)\}$ for $t \in \mathbb{R}$.

5.2 Use the moment generating function in question **5.1** to compute $\mathbb{E}(X)$ and $\mathbb{V}\text{ar}(X)$.

5.3 Suppose that $X_1, X_2, \ldots, X_n$ are independent with $X_i \sim \text{Po}(\lambda_i)$. Show that $Y = \sum_{i=1}^n X_i$ has a $\text{Po}(\mu)$ distribution, for a value of μ that you should determine.

5.4 Suppose that $Y_n \sim \text{Po}(n)$ and let

$$Z_n = \frac{Y_n - \mathbb{E}(Y_n)}{\sqrt{\mathbb{V}\text{ar}(Y_n)}}.$$

Let $M_{Z_n}(t)$ be the moment generating function of Z_n.
Use the moment generating function in question **5.1** to compute $\lim_{n\to\infty} M_{Z_n}(t)$. What does this tell you about the distribution of Z_n as $n \to \infty$?
You may use without proof the fact that the moment generating function M_Z of $Z \sim \mathcal{N}(0, 1)$ is $M_Z(t) = \mathrm{e}^{t^2/2}$.