Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2020	MATH2011-WE01

Title:
Complex Analysis II

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. Show your working and explain your reasoning.

Revision:	

Q1 **1.1** Show that a Möbius transformation that is not the identity has at most 2 fixed points in $\mathbb{C} \cup \{\infty\}$.

1.2 Suppose that $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$. Show that if $|a+d| < 2$, then the associated Möbius transformation M_T has exactly one fixed point in $\mathbb{H} = \{ z \in \mathbb{C} \, : \mathrm{Im}(z) > 0\}$.

1.3 State Rouché's Theorem.

1.4 Show that the polynomial $z^5 + 15z + 1$ has precisely four zeros (counted with multiplicity) in the set $\{ z : \frac{3}{2} \leq |z| < 2\}$.

Q2 **2.1** State the Cauchy-Riemann equations for a function $f : U \to \mathbb{C}$ on an open set $U \subset \mathbb{C}$, where $f = u + iv$ and u and v are real-valued functions on U. Prove that if f is complex differentiable on U then the Cauchy-Riemann equations hold at each point $z_0 \in U$.

2.2 Prove that if f is an entire function, and $g(z) = \overline{f(z)}$, then g is complex differentiable at z_0 if and only if $f'(z_0) = 0$.

2.3 Let f be an entire function. Prove that if $g(z) = \overline{f(z)}$ is also entire, then f is constant.

Q3 **3.1** Let (X, d_X) and (Y, d_Y) be two metric spaces. Prove that $f : X \to Y$ is continuous if and only if for every open set U in Y, $f^{-1}(U)$ is open in X.

3.2 Let $\mathrm{Arg} : \mathbb{C} - \{0\} \to \mathbb{R}$ be the principal value of the argument, i.e., taking values in $(-\pi, \pi]$. Using the standard metrics on $\mathbb{C} - \{0\}$ and $\mathbb{R}$, show that Arg is not continuous.

3.3 Let $\mathrm{Log} : \mathbb{C} - \{0\} \to \mathbb{C}$ be the principal branch of logarithm defined by

$$\mathrm{Log} = \log|z| + i\mathrm{Arg}(z).$$

Considering that $\mathrm{Arg}(z) = \mathrm{Im}(\mathrm{Log}(z))$, show that Log is not continuous with respect to the standard metrics on $\mathbb{C} - \{0\}$ and $\mathbb{C}$. You may use that $\mathrm{Im} : \mathbb{C} \to \mathbb{R}$ is a continuous function.

Q4 **4.1** State and prove Liouville's theorem. You may use Cauchy's Integral Formula in the proof.

4.2 Suppose that $f : \mathbb{C} \to \mathbb{C}$ is an entire function. Let $g(z) = f(\frac{1}{z})$. Prove that if g has a pole at 0 then there is $z_0 \in \mathbb{C}$ such that $f(z_0) = 0$. You may assume that g is holomorphic on $\mathbb{C} - \{0\}$.

Q5 **5.1** State Cauchy's residue theorem for simple closed contours.

5.2 Let $D_R(t) = \exp(\frac{2\pi i}{2020})(1-t)$, $t \in [0,1]$ and $L_R(t) = t$, $t \in [0,1]$. Let $f(z) = \frac{1}{z^{2020}+1}$. Show that

$$\int_{D_R} f(z)dz = -\exp\left(\frac{2\pi i}{2020}\right) \int_{L_R} f(z)dz$$

5.3 Let $C_R(z) = e^{2\pi i\theta}$, $\theta \in [0, \frac{1}{2020}]$. By integrating $f(z)$ along the wedge-shaped contour $\Gamma_R = L_R + C_R + D_R$, or otherwise, evaluate the integral

$$\int_0^\infty \frac{1}{x^{2020}+1} dx.$$