Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2020	MATH2031-WE01

Title: Analysis in Many Variables II

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. Show your working and explain your reasoning.

Revision:	

ED01/2020

Q1 **1.1** (i) Sketch the three-dimensional vector field

$$\mathbf{A}(x, y, z) = \frac{1}{2}(y-1)\mathbf{e}_1 + \frac{1}{2}(1-x)\mathbf{e}_2 + z^2\mathbf{e}_3$$

in the $z = 0$ plane.

(ii) Compute the curl of $\mathbf{A}$, comment on it in relation to your sketch, and discuss which components of $\mathbf{A}$ contribute to the curl and why.

(iii) Using index notation, calculate $\nabla \cdot (\mathbf{a} \times (\mathbf{x} \times \mathbf{a}))$, where $\mathbf{x}$ and $\mathbf{a}$ are 3-dimensional position and constant vectors respectively.

(iv) For f a scalar field in $\mathbb{R}^3$ and $\mathbf{v}$ a vector field in $\mathbb{R}^3$, use index notation to prove that

$$\nabla \times (f\mathbf{v}) \;=\; (\nabla f) \times \mathbf{v} + f\nabla \times \mathbf{v}$$

1.2 By integrating both sides against an arbitrary test-function, find coefficients a, b, c, d, g, h and k such that the following generalised function identities hold:

(i) $(x^2 - 6)\,\delta(x-6) = a\,\delta(x-b)\,;$

(ii) $(x-5)\,\delta''(x-5) = c\,\delta'(x-d)\,;$

(iii) $(x-4)\,\delta(x^2-4) = g\,\delta(x-k) + h\,\delta(x+k)\,.$

Q2 **2.1** Considering polar coordinates (r, θ) defined in terms of Cartesian coordinates (x, y) as usual by $x = r\cos\theta$ and $y = r\sin\theta$, show that the following four partial derivatives are given by:

$$\frac{\partial r}{\partial x} = \cos\theta\ , \qquad \frac{\partial r}{\partial y} = \sin\theta\ ,$$

$$\frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r}\ , \qquad \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}\ .$$

Hint: you may assume that $\frac{d}{du}(\tan^{-1}(u)) = \frac{1}{(1+u^2)}$.

2.2 If $f(x, y) = F(r, \theta)$ use the chain rule to obtain expressions for $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y^2}$ in terms of partial r and θ derivatives of F, and hence show that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \;=\; \frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}$$

2.3 Hence find the general rotationally-symmetric solution to

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \;=\; 0$$

which is non-singular away from the origin.

2.4 Find the specific rotationally-symmetric solution that satisfies $f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = 2$ and $f(0, e^2) = 4$.

Q3 **3.1** Consider the 2-dimensional scalar field $f(x, y)$. Write down the Taylor series for $f(\mathbf{a}+\mathbf{h})$ up to and including second order terms, around the point $\mathbf{a} = (x_0, y_0)$, where the vector $\mathbf{h} = (h_1, h_2)$ is assumed small.

3.2 If $\mathbf{a}$ is a critical point, show that the behaviour of the function around $\mathbf{a}$, as represented by $f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a})$, can be expressed, up to second order, purely in terms of $\mathbf{h} = (h_1, h_2)$ and the Hessian H, the elements of which you should define.

3.3 Find and classify all the critical points of

$$f(x, y) \;\; = \;\; xe^{-(x^2+y^2)}$$

3.4 Find the maxima and minima of $f(x, y)$ on the unit circle using the method of Lagrange multipliers.

3.5 Hence find the global maximum and minimum of $f(x, y)$ over the disc D defined as $\{D : x^2 + y^2 \leq 1\}$, stating clearly any results you use about the location of global extremum.

3.6 Consider the region $\{E : -1 \leq y \leq 1\}$. By relating the behaviour of $f(x, y)$ for points in $E\backslash D$ to that of points in D, discuss whether $f(x, y)$ will attain its global maximum and minimum over E.

Q4 **4.1** State Stokes' theorem, and use it to show that if $\mathbf{F}$ is a vector field in $\mathbb{R}^3$ and $\nabla \times \mathbf{F} = \mathbf{0}$ throughout a simply-connected region U, then the line integral $\int_C \mathbf{F} \cdot d\mathbf{x}$ of $\mathbf{F}$ between two points of U is independent of the path C between them.

4.2 State the divergence theorem, for a volume V bounded by a surface S.

Let V be the set of points between the paraboloid $z = x^2 + y^2$ and the plane $z = 2$ with positive x and y coordinates, so that

$$V = \{(x, y, z) : \; x \geq 0, y \geq 0, x^2 + y^2 \leq z \leq 2\}\,.$$

Find the volume of V, and then use the divergence theorem to compute

$$\int_{S_c} \mathbf{F} \cdot d\mathbf{A}$$

where $\mathbf{F}(x, y, z) = (3x{+}3)\,\mathbf{e}_1 + 2y\,\mathbf{e}_2 + z\,\mathbf{e}_3$, S_c is the curved surface of V, that is the set of points

$$S_c = \{(x, y, z) : \; z = x^2 + y^2, x \geq 0, y \geq 0, z \leq 2\}\,,$$

and the area elements $d\mathbf{A}$ point out of V. Note, S_c does not include the 'flat' parts of the surface of V, parallel to the $x - y$, $x - z$ and $y - z$ planes, so you'll have to compute these pieces separately and then subtract them from the divergence theorem result to get the quantity you're after.

Q5 **5.1** Show that

$$u(x,y) = xy$$

satisfies $\nabla^2 u = 0$ in the square $S = \{(x,y) \in \mathbb{R}^2 : 0 < x < 1, 0 < y < 1\}$. What Dirichlet boundary conditions does $u(x,y)$ satisfy on the four sides of S?

5.2 Show that

$$v(x,y) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) \sinh(n\pi y)\,,$$

satisfies the same equation as in part 5.1, namely $\nabla^2 v = 0$, in S, and the Dirichlet boundary conditions

$$v(0,y) = v(1,y) = 0, \quad v(x,0) = 0$$

on three of the four sides of S.

5.3 Find A_n given that $v(x,1) = x$ for $0 < x < 1$. The formula $\sin(A)\sin(B) = \frac{1}{2}(\cos(A-B) - \cos(A+B))$ can be used without proof.

5.4 If $v(x,y)$ is the solution to the problem posed in part 5.3, show that $w(x,y) = v(x,y) + v(y,x)$ solves the problem also solved in part 5.1.

5.5 Combine your results from parts 5.1 and 5.4 to show that for $0 < x < 1$,

$$\sum_{m=1}^{\infty} (-1)^{m+1} \frac{\sin(m\pi x)\sinh(m\pi x)}{m \sinh(m\pi)} = \frac{\pi x^2}{C}$$

where C is a constant which you should find. (Hint: consider $w(x,x)$; you can use, without proof, the fact that solutions to Laplace's equation in a finite domain with Dirichlet boundary conditions are unique.)