

## EXAMINATION PAPER

Examination Session: May/June

Year: 2020

Exam Code:

MATH2031-WE01

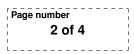
Title:

## Analysis in Many Variables II

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks.
	Please start each question on a new page. Please write your CIS username at the top of each page.
	Show your working and explain your reasoning.

**Revision:** 





Q1 1.1 (i) Sketch the three-dimensional vector field

$$\mathbf{A}(x, y, z) = \frac{1}{2}(y - 1)\mathbf{e}_1 + \frac{1}{2}(1 - x)\mathbf{e}_2 + z^2\mathbf{e}_3$$

in the z = 0 plane.

- (ii) Compute the curl of A, comment on it in relation to your sketch, and discuss which components of A contribute to the curl and why.
- (iii) Using index notation, calculate  $\nabla \cdot (\mathbf{a} \times (\mathbf{x} \times \mathbf{a}))$ , where  $\mathbf{x}$  and  $\mathbf{a}$  are 3-dimensional position and constant vectors respectively.
- (iv) For f a scalar field in  $\mathbb{R}^3$  and  $\mathbf{v}$  a vector field in  $\mathbb{R}^3$ , use index notation to prove that

$$\nabla \times (f\mathbf{v}) = (\nabla f) \times \mathbf{v} + f\nabla \times \mathbf{v}$$

- **1.2** By integrating both sides against an arbitrary test-function, find coefficients a, b, c, d, g, h and k such that the following generalised function identities hold:
  - (i)  $(x^2 6) \delta(x 6) = a \delta(x b);$
  - (ii)  $(x-5)\delta''(x-5) = c\delta'(x-d);$
  - (iii)  $(x-4)\delta(x^2-4) = g\delta(x-k) + h\delta(x+k)$ .
- **Q2 2.1** Considering polar coordinates  $(r, \theta)$  defined in terms of Cartesian coordinates (x, y) as usual by  $x = r \cos \theta$  and  $y = r \sin \theta$ , show that the following four partial derivatives are given by:

$$\frac{\partial r}{\partial x} = \cos \theta , \qquad \qquad \frac{\partial r}{\partial y} = \sin \theta ,$$
$$\frac{\partial \theta}{\partial x} = -\frac{\sin \theta}{r} , \qquad \qquad \frac{\partial \theta}{\partial y} = \frac{\cos \theta}{r} .$$

Hint: you may assume that  $\frac{d}{du}(\tan^{-1}(u)) = \frac{1}{(1+u^2)}$ .

**2.2** If  $f(x,y) = F(r,\theta)$  use the chain rule to obtain expressions for  $\frac{\partial^2 f}{\partial x^2}$  and  $\frac{\partial^2 f}{\partial y^2}$  in terms of partial r and  $\theta$  derivatives of F, and hence show that

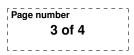
$$\frac{\partial^2 f}{\partial x^2} \ + \ \frac{\partial^2 f}{\partial y^2} \ = \ \frac{\partial^2 F}{\partial r^2} + \frac{1}{r} \frac{\partial F}{\partial r} + \frac{1}{r^2} \frac{\partial^2 F}{\partial \theta^2}$$

2.3 Hence find the general rotationally-symmetric solution to

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

which is non-singular away from the origin.

**2.4** Find the specific rotationally-symmetric solution that satisfies  $f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = 2$ and  $f(0, e^2) = 4$ .



- **Q3** 3.1 Consider the 2-dimensional scalar field f(x, y). Write down the Taylor series for  $f(\mathbf{a}+\mathbf{h})$  up to and including second order terms, around the point  $\mathbf{a} = (x_0, y_0)$ , where the vector  $\mathbf{h} = (h_1, h_2)$  is assumed small.
  - **3.2** If **a** is a critical point, show that the behaviour of the function around **a**, as represented by  $f(\mathbf{a} + \mathbf{h}) f(\mathbf{a})$ , can be expressed, up to second order, purely in terms of  $\mathbf{h} = (h_1, h_2)$  and the Hessian H, the elements of which you should define.
  - **3.3** Find and classify all the critical points of

$$f(x,y) = xe^{-(x^2+y^2)}$$

- **3.4** Find the maxima and minima of f(x, y) on the unit circle using the method of Lagrange multipliers.
- **3.5** Hence find the global maximum and minimum of f(x, y) over the disc D defined as  $\{D : x^2 + y^2 \le 1\}$ , stating clearly any results you use about the location of global extremum.
- **3.6** Consider the region  $\{E : -1 \le y \le 1\}$ . By relating the behaviour of f(x, y) for points in  $E \setminus D$  to that of points in D, discuss whether f(x, y) will attain its global maximum and minimum over E.
- **Q4** 4.1 State Stokes' theorem, and use it to show that if **F** is a vector field in  $\mathbb{R}^3$  and  $\nabla \times \mathbf{F} = \mathbf{0}$  throughout a simply-connected region U, then the line integral  $\int_C \mathbf{F} \cdot d\mathbf{x}$  of **F** between two points of U is independent of the path C between them.
  - 4.2 State the divergence theorem, for a volume V bounded by a surface S. Let V be the set of points between the paraboloid  $z = x^2 + y^2$  and the plane z = 2 with positive x and y coordinates, so that

$$V = \{(x, y, z) : x \ge 0, y \ge 0, x^2 + y^2 \le z \le 2\}.$$

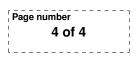
Find the volume of V, and then use the divergence theorem to compute

$$\int_{S_c} \mathbf{F} \cdot d\mathbf{A}$$

where  $\mathbf{F}(x, y, z) = (3x+3)\mathbf{e}_1 + 2y\mathbf{e}_2 + z\mathbf{e}_3$ ,  $S_c$  is the curved surface of V, that is the set of points

$$S_c = \{(x, y, z): z = x^2 + y^2, x \ge 0, y \ge 0, z \le 2\},\$$

and the area elements  $d\mathbf{A}$  point out of V. Note,  $S_c$  does not include the 'flat' parts of the surface of V, parallel to the x - y, x - z and y - z planes, so you'll have to compute these pieces separately and then subtract them from the divergence theorem result to get the quantity you're after.





**Q5** 5.1 Show that

$$u(x,y) = xy$$

satisfies  $\nabla^2 u = 0$  in the square  $S = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1, 0 < y < 1\}$ . What Dirichlet boundary conditions does u(x, y) satisfy on the four sides of S?

 ${\bf 5.2}$  Show that

$$v(x,y) = \sum_{n=1}^{\infty} A_n \, \sin(n\pi x) \, \sinh(n\pi y) \,,$$

satisfies the same equation as in part 5.1, namely  $\nabla^2 v = 0$ , in S, and the Dirichlet boundary conditions

$$v(0, y) = v(1, y) = 0, \quad v(x, 0) = 0$$

on three of the four sides of S.

- **5.3** Find  $A_n$  given that v(x, 1) = x for 0 < x < 1. The formula  $\sin(A)\sin(B) = \frac{1}{2}(\cos(A-B) \cos(A+B))$  can be used without proof.
- **5.4** If v(x, y) is the solution to the problem posed in part 5.3, show that w(x, y) = v(x, y) + v(y, x) solves the problem also solved in part 5.1.
- **5.5** Combine your results from parts 5.1 and 5.4 to show that for 0 < x < 1,

$$\sum_{m=1}^{\infty} (-1)^{m+1} \frac{\sin(m\pi x)\sinh(m\pi x)}{m\sinh(m\pi)} = \frac{\pi x^2}{C}$$

where C is a constant which you should find. (Hint: consider w(x, x); you can use, without proof, the fact that solutions to Laplace's equation in a finite domain with Dirichlet boundary conditions are unique.)