EXAMINATION PAPER

| Examination Session: | Year: | Exam Code: |
|---|---|---|
| May/June | 2020 | MATH2041-WE01 |

| Title: |
|---|
| Statistical Concepts II |

| | | |
|---|---|---|
| Time (for guidance only): | 3 hours | |
| Additional Material provided: | Tables: Normal distribution, t-distribution, chi-squared distribution, signed-rank test statistic, rank-sum test statistic. | |
| Materials Permitted: | | |
| Calculators Permitted: | Yes | Models Permitted: There is no restriction on the model of calculator which may be used. |

| | |
|---|---|
| Instructions to Candidates: | Credit will be given for your answers to all questions.<br>All questions carry the same marks.<br><br>Please start each question on a new page.<br>Please write your CIS username at the top of each page.<br><br>Show your working and explain your reasoning. |

| Revision: | |
|---|---|

ED01/2020

**Q1** **1.1** Suppose we wish to test a null hypothesis $H_0$ against an alternative hypothesis $H_1$, based on an observation $Y$ which takes values $1, 2, 3, 4, 5$. The probability distribution of $Y$ under each hypothesis is as follows.

| $y$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $\mathbb{P}[Y = y \mid H_0]$ | 0.05 | 0.1 | 0.15 | 0.3 | 0.4 |
| $\mathbb{P}[Y = y \mid H_1]$ | 0.15 | 0.25 | 0.3 | 0.2 | 0.1 |

(i) Suppose we seek a test which minimises the quantity

$$3\alpha + 2\beta,$$

where $\alpha$ is the probability of making a Type I error and $\beta$ is the probability of making a Type II error. Find the optimal test in this situation, and calculate the corresponding values of $\alpha$ and $\beta$. *You should state, without proof, any general results that you require in order to demonstrate that the test is optimal.*

(ii) An alternative version of this test is more concerned about the occurrence of a Type I error, and so places five times more weight on $\alpha$ than $\beta$. Explain briefly why the resulting test may not be useful for this problem.

(iii) For a fixed significance level of 0.15, find the most powerful test of these hypotheses and calculate the power of your test.

**1.2** The average consumption of cigarettes for a random sample of 9 smokers during each month before and after their enrolment in a quit-smoking mentorship program are recorded in the table below.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Before | 22 | 15 | 10 | 21 | 14 | 3 | 11 | 8 | 15 |
| After | 14 | 8 | 0 | 22 | 9 | 3 | 8 | 6 | 11 |

The researcher is interested in finding out whether the enrolment in a quit-smoking mentorship program causes a decline in cigarette smoking.

Use a Wilcoxon signed rank test to test this hypothesis at the 0.05 level of significance. Perform your test using:

(i) tables of exact critical values, and

(ii) using the normal approximation to the test statistic.

**Q2** **2.1** Show that for a general independent sample of size $n$, denoted $X_1, \ldots, X_n$, from a distribution $f(x \mid \theta)$ with unknown parameter $\theta$, the expected Fisher information for $\theta$ from the sample of $n$ observations, $\mathcal{I}_n(\theta)$, can be expressed as

$$\mathcal{I}_n(\theta) = n\mathcal{I}_1(\theta),$$

where $\mathcal{I}_1(\theta)$ is the expected Fisher information from a sample of size 1.

**2.2** The pdf for a random variable $Y \sim N(\mu, v)$ is

$$f(y \mid \mu, v) = \frac{1}{\sqrt{2\pi v}} e^{-\frac{1}{2v}(y-\mu)^2}.$$

Suppose that $X_1, \ldots, X_n$ form an independent sample from a normal distribution with mean 0 and variance $v$.

(i) Find a sufficient statistic for $v$ given data $(x_1, \ldots, x_n)$. Justify your answer.

(ii) Derive the maximum likelihood estimator $\hat{v}$ for $v$. Is $\hat{v}$ an unbiased estimator for $v$?

(iii) Using the sampling distribution for $\hat{v}$, derive an exact $100(1-\alpha)\%$ confidence interval for $v$ in terms of $\hat{v}$.

(iv) Find the expected Fisher information for $v$.

(v) Hence, for large $n$, calculate an approximate 95% confidence interval for $v$ in terms of $\hat{v}$. Clearly state each approximation that is made in order to construct the interval.

**Q3** The number of large meteorites that strike the Earth is monitored by a particular observatory. The observatory records the annual counts of meteorite strikes, $X$, which can be described by a Poisson distribution with an unknown parameter $\lambda$.

**3.1** Given a random sample $X_1, \ldots, X_{50}$ of $n = 50$ years of data, consider testing the null hypothesis $H_0 : \lambda = 2$ against the alternative hypothesis $H_1 : \lambda = 3$.

(i) Show that the optimal test of these hypotheses at significance level $\alpha$ can be expressed in the form "Reject $H_0$ if $T > k^*$", where $T = \sum_{i=1}^{n} X_i$ and $k^* > 0$ is a constant determined by $\alpha$ and the sampling distribution of $T$.

(ii) Find, approximately, the critical value $k^*$ for the test in part (i) using a significance level of 0.002, and find the approximate power of the test with this critical value.

**3.2** A further independent sample $Y_1, \ldots, Y_{50}$ of size 50 of meteorite impacts is recorded from another observatory using a more sensitive detector. $Y$ is also Poisson, but with an unknown parameter $\mu$, where it is believed that $2\mu = 3\lambda$ to reflect the increased sensitivity.

Using both samples and assuming $2\mu = 3\lambda$:

(i) Find the maximum likelihood estimator for $\lambda$.

(ii) Show that the generalised likelihood ratio test statistic for the hypotheses $H_0 : \lambda = 2$ against the alternative $H_1 : \lambda \neq 2$ is of the form:

$$e^{S-250} \left( \frac{250}{S} \right)^{S},$$

where $S = \sum_{i=1}^{50} X_i + \sum_{j=1}^{50} Y_j$.

(iii) Hence find a simplified expression for the approximate large-sample generalised likelihood ratio test statistic for these hypotheses, and the corresponding critical value at significance level 0.05.

**Q4** The beta distribution, with parameters $a > 0, b > 0$ has probability density function

$$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1},\ 0 < x < 1.$$

**4.1** Show that if the random quantity $X$ has a beta distribution, with parameters $a, b$, then

$$E(X) = \frac{a}{a+b},\ \ Var(X) = \frac{ab}{(a+b)^2(a+b+1)}.$$

**4.2** Explain what it means to say that a family of probability distributions is a conjugate prior family for sampling from a particular family of distributions. Show that the family of beta distributions is a conjugate prior family for sampling from the binomial distribution.

**4.3** Suppose that a particular medical treatment has a probability $q$ of success for a randomly selected patient with a particular disease. Suppose that the prior distribution for $q$ is a beta distribution with parameters $a = b = 4$. Suppose that, in a random sample of 60 patients, the treatment is found to be successful for 36 of the patients. Find the posterior probability distribution for $q$. Therefore, find the probability that the next two patients that are treated will both have successful outcomes.

**4.4** Explain what is meant by an $\alpha$ level credible interval for $q$. Evaluate an approximate posterior credible interval for $q$, in the problem of question **4.3**, at level 0.95.

**4.5** State the form of the limiting posterior distribution for sampling from a general likelihood as the sample size increases. Find the limiting posterior form for the problem of question **4.3**. For the given data, compare the mean and standard deviation of this approximation to the exact values for these quantities for the posterior distribution calculated in question **4.3**.

**Q5** The 4 C's (cut, colour, clarity, and carat weight) are the main criteria used to judge the quality of a diamond and determine its value (price). In order to understand how carat weight affects the price of a diamond, a random sample of 20 diamonds were obtained, and their carat weight and price (in dollars) are given below.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| carat | 1.02 | 1.54 | 0.66 | 0.37 | 1.08 | 0.72 | 0.53 | 0.31 | 0.71 | 0.41 |
| price | 5666 | 9562 | 2648 | 791 | 4685 | 2843 | 1948 | 680 | 2946 | 964 |
| carat | 0.91 | 0.9 | 0.5 | 1 | 1.37 | 0.33 | 0.53 | 0.64 | 0.48 | 0.54 |
| price | 3639 | 4229 | 963 | 6272 | 8577 | 594 | 1678 | 4281 | 990 | 1754 |

**5.1** Consider the simple linear regression model for the price on carat weight, $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$. Let the least-squares regression line be $y = \hat{\beta}_0 + \hat{\beta}_1 x$.

(i) State the assumptions of the usual simple linear regression model with normal errors.

(ii) Give expressions for $\hat{\beta}_0$ and $\hat{\beta}_1$. Prove that they are unbiased estimators of $\beta_0$ and $\beta_1$.

**5.2** A simple linear regression model was fitted to the above data using `R`, as shown on the next page. Use the data and the `R` output to answer the following.

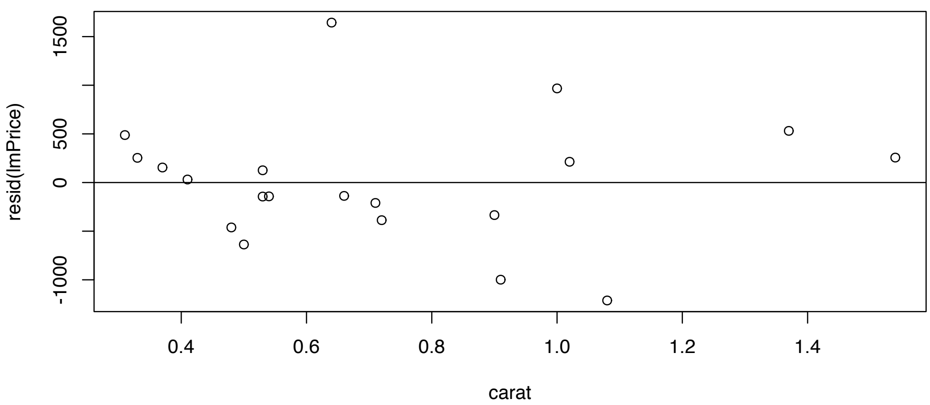
(i) What are the estimates of the intercept and slope for the above data? Interpret the t-values for the coefficients. Give a 95% confidence interval for the slope.

(ii) Explain how the 'Multiple R-squared' value is calculated, and interpret the corresponding value for this regression from the `R` output.

(iii) For a diamond with carat weight $x^* = 0.75$, estimate the expected value of price, $y$, and give a 95% confidence interval for the expected value.

(iv) Explain how the residuals are calculated. Use the residual plot to assess informally the adequacy of the linear regression model as a summary description of these data.

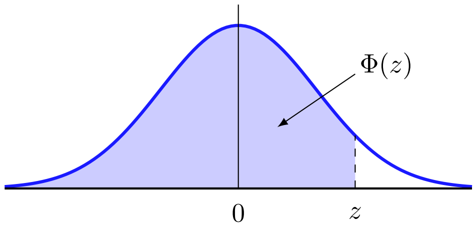
```
> lmPrice = lm(price~carat, data=diamonds1)
> summary(lmPrice)

Call:
lm(formula = price ~ carat, data = diamonds1)

Residuals:
     Min       1Q   Median       3Q      Max
-1212.31  -347.70   -53.19   254.39  1643.82

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  -2104.8      354.1  -5.944 1.26e-05 ***
carat         7409.4      442.6  16.742 2.02e-12 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 659.4 on 18 degrees of freedom
Multiple R-squared:  0.9397,Adjusted R-squared:  0.9363
F-statistic: 280.3 on 1 and 18 DF,  p-value: 2.023e-12

> mean(carat)
[1] 0.7275
> var(carat)
[1] 0.1168303
> plot(resid(lmPrice)~carat)
> abline(0,0)
```



# Table A: Probabilities for the standard normal distribution

Table entry for $z$ is the probability lying to the left of $z$ for a $N(0,1)$ distribution.

$$\Phi(z) = P\left[Z < z\right]$$

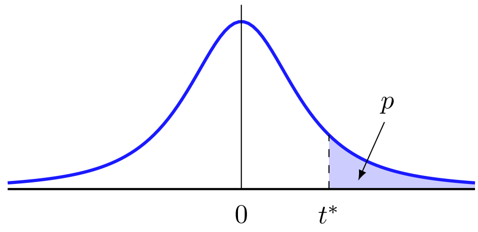


$N(0,1)$ density

| $z$ | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.00 | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0.5279 | 0.5319 | 0.5359 |
| 0.10 | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0.5675 | 0.5714 | 0.5753 |
| 0.20 | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 | 0.5987 | 0.6026 | 0.6064 | 0.6103 | 0.6141 |
| 0.30 | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0.6443 | 0.6480 | 0.6517 |
| 0.40 | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | 0.6844 | 0.6879 |
| 0.50 | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | 0.7190 | 0.7224 |
| 0.60 | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | 0.7517 | 0.7549 |
| 0.70 | 0.7580 | 0.7611 | 0.7642 | 0.7673 | 0.7704 | 0.7734 | 0.7764 | 0.7794 | 0.7823 | 0.7852 |
| 0.80 | 0.7881 | 0.7910 | 0.7939 | 0.7967 | 0.7995 | 0.8023 | 0.8051 | 0.8078 | 0.8106 | 0.8133 |
| 0.90 | 0.8159 | 0.8186 | 0.8212 | 0.8238 | 0.8264 | 0.8289 | 0.8315 | 0.8340 | 0.8365 | 0.8389 |
| 1.00 | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.8621 |
| 1.10 | 0.8643 | 0.8665 | 0.8686 | 0.8708 | 0.8729 | 0.8749 | 0.8770 | 0.8790 | 0.8810 | 0.8830 |
| 1.20 | 0.8849 | 0.8869 | 0.8888 | 0.8907 | 0.8925 | 0.8944 | 0.8962 | 0.8980 | 0.8997 | 0.9015 |
| 1.30 | 0.9032 | 0.9049 | 0.9066 | 0.9082 | 0.9099 | 0.9115 | 0.9131 | 0.9147 | 0.9162 | 0.9177 |
| 1.40 | 0.9192 | 0.9207 | 0.9222 | 0.9236 | 0.9251 | 0.9265 | 0.9279 | 0.9292 | 0.9306 | 0.9319 |
| 1.50 | 0.9332 | 0.9345 | 0.9357 | 0.9370 | 0.9382 | 0.9394 | 0.9406 | 0.9418 | 0.9429 | 0.9441 |
| 1.60 | 0.9452 | 0.9463 | 0.9474 | 0.9484 | 0.9495 | 0.9505 | 0.9515 | 0.9525 | 0.9535 | 0.9545 |
| 1.70 | 0.9554 | 0.9564 | 0.9573 | 0.9582 | 0.9591 | 0.9599 | 0.9608 | 0.9616 | 0.9625 | 0.9633 |
| 1.80 | 0.9641 | 0.9649 | 0.9656 | 0.9664 | 0.9671 | 0.9678 | 0.9686 | 0.9693 | 0.9699 | 0.9706 |
| 1.90 | 0.9713 | 0.9719 | 0.9726 | 0.9732 | 0.9738 | 0.9744 | 0.9750 | 0.9756 | 0.9761 | 0.9767 |
| 2.00 | 0.9772 | 0.9778 | 0.9783 | 0.9788 | 0.9793 | 0.9798 | 0.9803 | 0.9808 | 0.9812 | 0.9817 |
| 2.10 | 0.9821 | 0.9826 | 0.9830 | 0.9834 | 0.9838 | 0.9842 | 0.9846 | 0.9850 | 0.9854 | 0.9857 |
| 2.20 | 0.9861 | 0.9864 | 0.9868 | 0.9871 | 0.9875 | 0.9878 | 0.9881 | 0.9884 | 0.9887 | 0.9890 |
| 2.30 | 0.9893 | 0.9896 | 0.9898 | 0.9901 | 0.9904 | 0.9906 | 0.9909 | 0.9911 | 0.9913 | 0.9916 |
| 2.40 | 0.9918 | 0.9920 | 0.9922 | 0.9925 | 0.9927 | 0.9929 | 0.9931 | 0.9932 | 0.9934 | 0.9936 |
| 2.50 | 0.9938 | 0.9940 | 0.9941 | 0.9943 | 0.9945 | 0.9946 | 0.9948 | 0.9949 | 0.9951 | 0.9952 |
| 2.60 | 0.9953 | 0.9955 | 0.9956 | 0.9957 | 0.9959 | 0.9960 | 0.9961 | 0.9962 | 0.9963 | 0.9964 |
| 2.70 | 0.9965 | 0.9966 | 0.9967 | 0.9968 | 0.9969 | 0.9970 | 0.9971 | 0.9972 | 0.9973 | 0.9974 |
| 2.80 | 0.9974 | 0.9975 | 0.9976 | 0.9977 | 0.9977 | 0.9978 | 0.9979 | 0.9979 | 0.9980 | 0.9981 |
| 2.90 | 0.9981 | 0.9982 | 0.9982 | 0.9983 | 0.9984 | 0.9984 | 0.9985 | 0.9985 | 0.9986 | 0.9986 |
| 3.00 | 0.9987 | 0.9987 | 0.9987 | 0.9988 | 0.9988 | 0.9989 | 0.9989 | 0.9989 | 0.9990 | 0.9990 |
| 3.10 | 0.9990 | 0.9991 | 0.9991 | 0.9991 | 0.9992 | 0.9992 | 0.9992 | 0.9992 | 0.9993 | 0.9993 |
| 3.20 | 0.9993 | 0.9993 | 0.9994 | 0.9994 | 0.9994 | 0.9994 | 0.9994 | 0.9995 | 0.9995 | 0.9995 |
| 3.30 | 0.9995 | 0.9995 | 0.9995 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9997 |
| 3.40 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9998 |

# Table B: Probabilities for the $t$-distribution

Table entry for $p$ and $C$ is the critical value $t^*$ with probability $p$ lying to its right and probability $C$ lying between $-t^*$ and $t^*$ for a $t$-distribution with $k$ degrees of freedom.

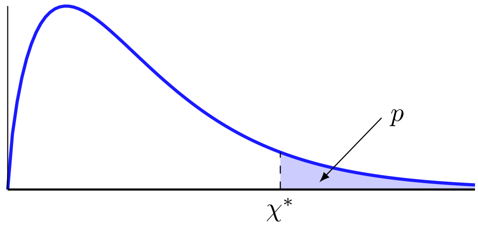



$t_k$ density

| | Upper-tail probability $p$ | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $k$ | 0.25 | 0.2 | 0.15 | 0.1 | 0.05 | 0.025 | 0.02 | 0.01 | 0.005 | 0.0025 | 0.001 | 0.0005 |
| 1 | 1.000 | 1.376 | 1.963 | 3.078 | 6.314 | 12.706 | 15.895 | 31.821 | 63.657 | 127.321 | 318.309 | 636.619 |
| 2 | 0.816 | 1.061 | 1.386 | 1.886 | 2.920 | 4.303 | 4.849 | 6.965 | 9.925 | 14.089 | 22.327 | 31.599 |
| 3 | 0.765 | 0.978 | 1.250 | 1.638 | 2.353 | 3.182 | 3.482 | 4.541 | 5.841 | 7.453 | 10.215 | 12.924 |
| 4 | 0.741 | 0.941 | 1.190 | 1.533 | 2.132 | 2.776 | 2.999 | 3.747 | 4.604 | 5.598 | 7.173 | 8.610 |
| 5 | 0.727 | 0.920 | 1.156 | 1.476 | 2.015 | 2.571 | 2.757 | 3.365 | 4.032 | 4.773 | 5.893 | 6.869 |
| 6 | 0.718 | 0.906 | 1.134 | 1.440 | 1.943 | 2.447 | 2.612 | 3.143 | 3.707 | 4.317 | 5.208 | 5.959 |
| 7 | 0.711 | 0.896 | 1.119 | 1.415 | 1.895 | 2.365 | 2.517 | 2.998 | 3.499 | 4.029 | 4.785 | 5.408 |
| 8 | 0.706 | 0.889 | 1.108 | 1.397 | 1.860 | 2.306 | 2.449 | 2.896 | 3.355 | 3.833 | 4.501 | 5.041 |
| 9 | 0.703 | 0.883 | 1.100 | 1.383 | 1.833 | 2.262 | 2.398 | 2.821 | 3.250 | 3.690 | 4.297 | 4.781 |
| 10 | 0.700 | 0.879 | 1.093 | 1.372 | 1.812 | 2.228 | 2.359 | 2.764 | 3.169 | 3.581 | 4.144 | 4.587 |
| 11 | 0.697 | 0.876 | 1.088 | 1.363 | 1.796 | 2.201 | 2.328 | 2.718 | 3.106 | 3.497 | 4.025 | 4.437 |
| 12 | 0.695 | 0.873 | 1.083 | 1.356 | 1.782 | 2.179 | 2.303 | 2.681 | 3.055 | 3.428 | 3.930 | 4.318 |
| 13 | 0.694 | 0.870 | 1.079 | 1.350 | 1.771 | 2.160 | 2.282 | 2.650 | 3.012 | 3.372 | 3.852 | 4.221 |
| 14 | 0.692 | 0.868 | 1.076 | 1.345 | 1.761 | 2.145 | 2.264 | 2.624 | 2.977 | 3.326 | 3.787 | 4.140 |
| 15 | 0.691 | 0.866 | 1.074 | 1.341 | 1.753 | 2.131 | 2.249 | 2.602 | 2.947 | 3.286 | 3.733 | 4.073 |
| 16 | 0.690 | 0.865 | 1.071 | 1.337 | 1.746 | 2.120 | 2.235 | 2.583 | 2.921 | 3.252 | 3.686 | 4.015 |
| 17 | 0.689 | 0.863 | 1.069 | 1.333 | 1.740 | 2.110 | 2.224 | 2.567 | 2.898 | 3.222 | 3.646 | 3.965 |
| 18 | 0.688 | 0.862 | 1.067 | 1.330 | 1.734 | 2.101 | 2.214 | 2.552 | 2.878 | 3.197 | 3.610 | 3.922 |
| 19 | 0.688 | 0.861 | 1.066 | 1.328 | 1.729 | 2.093 | 2.205 | 2.539 | 2.861 | 3.174 | 3.579 | 3.883 |
| 20 | 0.687 | 0.860 | 1.064 | 1.325 | 1.725 | 2.086 | 2.197 | 2.528 | 2.845 | 3.153 | 3.552 | 3.850 |
| 21 | 0.686 | 0.859 | 1.063 | 1.323 | 1.721 | 2.080 | 2.189 | 2.518 | 2.831 | 3.135 | 3.527 | 3.819 |
| 22 | 0.686 | 0.858 | 1.061 | 1.321 | 1.717 | 2.074 | 2.183 | 2.508 | 2.819 | 3.119 | 3.505 | 3.792 |
| 23 | 0.685 | 0.858 | 1.060 | 1.319 | 1.714 | 2.069 | 2.177 | 2.500 | 2.807 | 3.104 | 3.485 | 3.768 |
| 24 | 0.685 | 0.857 | 1.059 | 1.318 | 1.711 | 2.064 | 2.172 | 2.492 | 2.797 | 3.091 | 3.467 | 3.745 |
| 25 | 0.684 | 0.856 | 1.058 | 1.316 | 1.708 | 2.060 | 2.167 | 2.485 | 2.787 | 3.078 | 3.450 | 3.725 |
| 26 | 0.684 | 0.856 | 1.058 | 1.315 | 1.706 | 2.056 | 2.162 | 2.479 | 2.779 | 3.067 | 3.435 | 3.707 |
| 27 | 0.684 | 0.855 | 1.057 | 1.314 | 1.703 | 2.052 | 2.158 | 2.473 | 2.771 | 3.057 | 3.421 | 3.690 |
| 28 | 0.683 | 0.855 | 1.056 | 1.313 | 1.701 | 2.048 | 2.154 | 2.467 | 2.763 | 3.047 | 3.408 | 3.674 |
| 29 | 0.683 | 0.854 | 1.055 | 1.311 | 1.699 | 2.045 | 2.150 | 2.462 | 2.756 | 3.038 | 3.396 | 3.659 |
| 30 | 0.683 | 0.854 | 1.055 | 1.310 | 1.697 | 2.042 | 2.147 | 2.457 | 2.750 | 3.030 | 3.385 | 3.646 |
| 40 | 0.681 | 0.851 | 1.050 | 1.303 | 1.684 | 2.021 | 2.123 | 2.423 | 2.704 | 2.971 | 3.307 | 3.551 |
| 50 | 0.679 | 0.849 | 1.047 | 1.299 | 1.676 | 2.009 | 2.109 | 2.403 | 2.678 | 2.937 | 3.261 | 3.496 |
| 60 | 0.679 | 0.848 | 1.045 | 1.296 | 1.671 | 2.000 | 2.099 | 2.390 | 2.660 | 2.915 | 3.232 | 3.460 |
| 70 | 0.678 | 0.847 | 1.044 | 1.294 | 1.667 | 1.994 | 2.093 | 2.381 | 2.648 | 2.899 | 3.211 | 3.435 |
| 100 | 0.677 | 0.845 | 1.042 | 1.290 | 1.660 | 1.984 | 2.081 | 2.364 | 2.626 | 2.871 | 3.174 | 3.390 |
| 1000 | 0.675 | 0.842 | 1.037 | 1.282 | 1.646 | 1.962 | 2.056 | 2.330 | 2.581 | 2.813 | 3.098 | 3.300 |
| $\infty$ | 0.674 | 0.842 | 1.036 | 1.282 | 1.645 | 1.960 | 2.054 | 2.326 | 2.576 | 2.807 | 3.090 | 3.291 |
| | 50% | 60% | 70% | 80% | 90% | 95% | 96% | 98% | 99% | 99.5% | 99.8% | 99.9% |
| | Confidence level $C$ | | | | | | | | | | | |

# Table C: Probabilities for the $\chi^2$-distribution

Table entry for $p$ is the point $\chi^*$ with probability $p$ lying above it for a $\chi^2$ distribution with $k$ degrees of freedom.




$\chi^2_k$ density

| | Upper-tail probability $p$ | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $k$ | 0.995 | 0.99 | 0.975 | 0.95 | 0.9 | 0.75 | 0.25 | 0.1 | 0.05 | 0.025 | 0.01 | 0.005 |
| 1 | 0.000 | 0.000 | 0.001 | 0.004 | 0.016 | 0.102 | 1.323 | 2.706 | 3.841 | 5.024 | 6.635 | 7.879 |
| 2 | 0.010 | 0.020 | 0.051 | 0.103 | 0.211 | 0.575 | 2.773 | 4.605 | 5.991 | 7.378 | 9.210 | 10.597 |
| 3 | 0.072 | 0.115 | 0.216 | 0.352 | 0.584 | 1.213 | 4.108 | 6.251 | 7.815 | 9.348 | 11.345 | 12.838 |
| 4 | 0.207 | 0.297 | 0.484 | 0.711 | 1.064 | 1.923 | 5.385 | 7.779 | 9.488 | 11.143 | 13.277 | 14.860 |
| 5 | 0.412 | 0.554 | 0.831 | 1.145 | 1.610 | 2.675 | 6.626 | 9.236 | 11.070 | 12.833 | 15.086 | 16.750 |
| 6 | 0.676 | 0.872 | 1.237 | 1.635 | 2.204 | 3.455 | 7.841 | 10.645 | 12.592 | 14.449 | 16.812 | 18.548 |
| 7 | 0.989 | 1.239 | 1.690 | 2.167 | 2.833 | 4.255 | 9.037 | 12.017 | 14.067 | 16.013 | 18.475 | 20.278 |
| 8 | 1.344 | 1.646 | 2.180 | 2.733 | 3.490 | 5.071 | 10.219 | 13.362 | 15.507 | 17.535 | 20.090 | 21.955 |
| 9 | 1.735 | 2.088 | 2.700 | 3.325 | 4.168 | 5.899 | 11.389 | 14.684 | 16.919 | 19.023 | 21.666 | 23.589 |
| 10 | 2.156 | 2.558 | 3.247 | 3.940 | 4.865 | 6.737 | 12.549 | 15.987 | 18.307 | 20.483 | 23.209 | 25.188 |
| 11 | 2.603 | 3.053 | 3.816 | 4.575 | 5.578 | 7.584 | 13.701 | 17.275 | 19.675 | 21.920 | 24.725 | 26.757 |
| 12 | 3.074 | 3.571 | 4.404 | 5.226 | 6.304 | 8.438 | 14.845 | 18.549 | 21.026 | 23.337 | 26.217 | 28.300 |
| 13 | 3.565 | 4.107 | 5.009 | 5.892 | 7.042 | 9.299 | 15.984 | 19.812 | 22.362 | 24.736 | 27.688 | 29.819 |
| 14 | 4.075 | 4.660 | 5.629 | 6.571 | 7.790 | 10.165 | 17.117 | 21.064 | 23.685 | 26.119 | 29.141 | 31.319 |
| 15 | 4.601 | 5.229 | 6.262 | 7.261 | 8.547 | 11.037 | 18.245 | 22.307 | 24.996 | 27.488 | 30.578 | 32.801 |
| 16 | 5.142 | 5.812 | 6.908 | 7.962 | 9.312 | 11.912 | 19.369 | 23.542 | 26.296 | 28.845 | 32.000 | 34.267 |
| 17 | 5.697 | 6.408 | 7.564 | 8.672 | 10.085 | 12.792 | 20.489 | 24.769 | 27.587 | 30.191 | 33.409 | 35.718 |
| 18 | 6.265 | 7.015 | 8.231 | 9.390 | 10.865 | 13.675 | 21.605 | 25.989 | 28.869 | 31.526 | 34.805 | 37.156 |
| 19 | 6.844 | 7.633 | 8.907 | 10.117 | 11.651 | 14.562 | 22.718 | 27.204 | 30.144 | 32.852 | 36.191 | 38.582 |
| 20 | 7.434 | 8.260 | 9.591 | 10.851 | 12.443 | 15.452 | 23.828 | 28.412 | 31.410 | 34.170 | 37.566 | 39.997 |
| 21 | 8.034 | 8.897 | 10.283 | 11.591 | 13.240 | 16.344 | 24.935 | 29.615 | 32.671 | 35.479 | 38.932 | 41.401 |
| 22 | 8.643 | 9.542 | 10.982 | 12.338 | 14.041 | 17.240 | 26.039 | 30.813 | 33.924 | 36.781 | 40.289 | 42.796 |
| 23 | 9.260 | 10.196 | 11.689 | 13.091 | 14.848 | 18.137 | 27.141 | 32.007 | 35.172 | 38.076 | 41.638 | 44.181 |
| 24 | 9.886 | 10.856 | 12.401 | 13.848 | 15.659 | 19.037 | 28.241 | 33.196 | 36.415 | 39.364 | 42.980 | 45.559 |
| 25 | 10.520 | 11.524 | 13.120 | 14.611 | 16.473 | 19.939 | 29.339 | 34.382 | 37.652 | 40.646 | 44.314 | 46.928 |
| 26 | 11.160 | 12.198 | 13.844 | 15.379 | 17.292 | 20.843 | 30.435 | 35.563 | 38.885 | 41.923 | 45.642 | 48.290 |
| 27 | 11.808 | 12.879 | 14.573 | 16.151 | 18.114 | 21.749 | 31.528 | 36.741 | 40.113 | 43.195 | 46.963 | 49.645 |
| 28 | 12.461 | 13.565 | 15.308 | 16.928 | 18.939 | 22.657 | 32.620 | 37.916 | 41.337 | 44.461 | 48.278 | 50.993 |
| 29 | 13.121 | 14.256 | 16.047 | 17.708 | 19.768 | 23.567 | 33.711 | 39.087 | 42.557 | 45.722 | 49.588 | 52.336 |
| 30 | 13.787 | 14.953 | 16.791 | 18.493 | 20.599 | 24.478 | 34.800 | 40.256 | 43.773 | 46.979 | 50.892 | 53.672 |
| 40 | 20.707 | 22.164 | 24.433 | 26.509 | 29.051 | 33.660 | 45.616 | 51.805 | 55.758 | 59.342 | 63.691 | 66.766 |
| 50 | 27.991 | 29.707 | 32.357 | 34.764 | 37.689 | 42.942 | 56.334 | 63.167 | 67.505 | 71.420 | 76.154 | 79.490 |
| 60 | 35.534 | 37.485 | 40.482 | 43.188 | 46.459 | 52.294 | 66.981 | 74.397 | 79.082 | 83.298 | 88.379 | 91.952 |
| 70 | 43.275 | 45.442 | 48.758 | 51.739 | 55.329 | 61.698 | 77.577 | 85.527 | 90.531 | 95.023 | 100.425 | 104.215 |
| 80 | 51.172 | 53.540 | 57.153 | 60.391 | 64.278 | 71.145 | 88.130 | 96.578 | 101.879 | 106.629 | 112.329 | 116.321 |
| 100 | 67.328 | 70.065 | 74.222 | 77.929 | 82.358 | 90.133 | 109.141 | 118.498 | 124.342 | 129.561 | 135.807 | 140.169 |
| | 0.005 | 0.01 | 0.025 | 0.05 | 0.1 | 0.25 | 0.75 | 0.9 | 0.95 | 0.975 | 0.99 | 0.995 |
| | Lower-tail probability $\alpha$ | | | | | | | | | | | |

# Table D: Critical values for the Rank-sum Test

Reject the hypothesis of identical populations in a two-sided test at level $\alpha$ if the test statistic $W$ from a group of size $n$ is *less than* the value $T_L$ shown in the following table, or greater than the value $T_U$ where

$$T_U = n(n+m+1) - T_L.$$

| | | $m$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\alpha = 0.01$ | | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| | 2 | - | - | - | - | - | - | - | - | - |
| | 3 | - | - | - | - | - | - | - | 7 | 7 |
| | 4 | - | - | - | - | 11 | 11 | 12 | 12 | 13 |
| | 5 | - | - | - | 16 | 17 | 17 | 18 | 19 | 20 |
| $n$ | 6 | - | - | 22 | 23 | 24 | 25 | 26 | 27 | 28 |
| | 7 | - | - | 29 | 30 | 32 | 33 | 35 | 36 | 38 |
| | 8 | - | - | 38 | 39 | 41 | 43 | 44 | 46 | 48 |
| | 9 | - | 46 | 47 | 49 | 51 | 53 | 55 | 57 | 59 |
| | 10 | - | 56 | 58 | 60 | 62 | 65 | 67 | 69 | 72 |

| | | $m$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\alpha = 0.05$ | | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| | 2 | - | - | - | - | - | - | 4 | 4 | 4 |
| | 3 | - | - | - | 7 | 8 | 8 | 9 | 9 | 10 |
| | 4 | - | - | 11 | 12 | 13 | 14 | 15 | 15 | 16 |
| | 5 | - | 16 | 17 | 18 | 19 | 21 | 22 | 23 | 24 |
| $n$ | 6 | - | 23 | 24 | 25 | 27 | 28 | 30 | 32 | 33 |
| | 7 | - | 30 | 32 | 34 | 35 | 37 | 39 | 41 | 43 |
| | 8 | 37 | 39 | 41 | 43 | 45 | 47 | 50 | 52 | 54 |
| | 9 | 46 | 48 | 50 | 53 | 56 | 58 | 61 | 63 | 66 |
| | 10 | 56 | 59 | 61 | 64 | 67 | 70 | 73 | 76 | 79 |

| | | $m$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\alpha = 0.10$ | | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| | 2 | - | - | - | 4 | 4 | 4 | 5 | 5 | 5 |
| | 3 | - | 7 | 7 | 8 | 9 | 9 | 10 | 11 | 11 |
| | 4 | - | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
| | 5 | 16 | 17 | 18 | 20 | 21 | 22 | 24 | 25 | 27 |
| $n$ | 6 | 22 | 24 | 25 | 27 | 29 | 30 | 32 | 34 | 36 |
| | 7 | 29 | 31 | 33 | 35 | 37 | 40 | 42 | 44 | 46 |
| | 8 | 38 | 40 | 42 | 45 | 47 | 50 | 52 | 55 | 57 |
| | 9 | 47 | 50 | 52 | 55 | 58 | 61 | 64 | 67 | 70 |
| | 10 | 57 | 60 | 63 | 67 | 70 | 73 | 76 | 80 | 83 |

# Table E: Critical values for the Signed-rank Test

Reject the null hypothesis of identical populations if the test statistic $V$ is *less than* the value $T$ shown in the following table.

| Sample size | Level of significance for a two-sided test | | | | | |
|---|---|---|---|---|---|---|
| $n$ | 20% | 10% | 5% | 2% | 1% | 0.5% |
| | 10% | 5% | 2.5% | 1% | 0.5% | 0.25% |
| | Level of significance for a one-sided test | | | | | |
| 5 | 3 | 1 | - | - | - | - |
| 6 | 4 | 3 | 1 | - | - | - |
| 7 | 6 | 4 | 3 | 1 | - | - |
| 8 | 9 | 6 | 4 | 2 | 1 | - |
| 9 | 11 | 9 | 6 | 4 | 2 | 1 |
| 10 | 15 | 11 | 9 | 6 | 4 | 2 |
| 11 | 18 | 14 | 11 | 8 | 6 | 4 |
| 12 | 22 | 18 | 14 | 10 | 8 | 6 |
| 13 | 27 | 22 | 18 | 13 | 10 | 8 |
| 14 | 32 | 26 | 22 | 16 | 13 | 10 |
| 15 | 37 | 31 | 26 | 20 | 16 | 13 |
| 16 | 43 | 36 | 30 | 24 | 20 | 16 |
| 17 | 49 | 42 | 35 | 28 | 24 | 20 |
| 18 | 56 | 48 | 41 | 33 | 28 | 24 |
| 19 | 63 | 54 | 47 | 38 | 33 | 28 |
| 20 | 70 | 61 | 53 | 44 | 38 | 33 |
| 21 | 78 | 68 | 59 | 50 | 43 | 38 |
| 22 | 87 | 76 | 66 | 56 | 49 | 43 |
| 23 | 95 | 84 | 74 | 63 | 55 | 49 |
| 24 | 105 | 92 | 82 | 70 | 62 | 55 |
| 25 | 114 | 101 | 90 | 77 | 69 | 61 |
| 26 | 125 | 111 | 99 | 85 | 76 | 68 |
| 27 | 135 | 120 | 108 | 93 | 84 | 75 |
| 28 | 146 | 131 | 117 | 102 | 92 | 83 |
| 29 | 158 | 141 | 127 | 111 | 101 | 91 |
| 30 | 170 | 152 | 138 | 121 | 110 | 99 |