

EXAMINATION PAPER

Examination Session: May/June

2020

Year:

Exam Code:

MATH2051-WE01

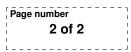
Title:

Numerical Analysis II

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks.
	Please start each question on a new page. Please write your CIS username at the top of each page.
	Show your working and explain your reasoning.

Revision:





- **Q1 1.1** Explain what it means for a floating-point number to have (i) a *finite precision* and (ii) a *finite range*.
 - **1.2** A student tried to use bisection in Python to solve $f_1(x) = x 1 = 0$ and $f_3(x) = x^3 3x^2 + 3x 1 = 0$, both with initial interval [-1, 2]. With f_1 he obtained 1.0 (the exact answer), while with f_3 he obtained 1.0000038146972656. Explain why f_3 gives a worse result, and how it relates to **1.1**.
 - **1.3** Seeking instead to find the roots of $f_1(x) = x$ and $f_3(x) = x^3$, should we expect similar results (in terms of accuracy of the answer)? Justify your answer briefly.
- **Q2 2.1** Determine the orders of convergence of the following sequences (all converging to 0): $x_n = (n!)^{-1}$, $y_n = 2^{-3^n}$ and $z_n = 5^{-n^2}$. Justify your answers briefly.
 - **2.2** Show that the iteration $x_{k+1} = \cos x_k$ is a contraction mapping in some interval $I \subset [\frac{1}{2}, 1]$, and give one such interval.
 - **2.3** Show that the same iteration $x_{k+1} = \cos x_k$ is in fact convergent for all $x_0 \in \mathbb{R}$.
- Q3 Given $f(x) = 7\cos^2(\pi x/2)$, we seek a $p \in \mathcal{P}_4$ such that p(-1) = f(-1), p'(-1) = f'(-1), p(0) = f(0), p(1) = f(1) and p'(1) = f'(1).
 - **3.1** Compute the divided difference table using the usual Newton interpolation construction, i.e. with $[a, a]f := \lim_{y \to a} [a, y]f = f'(a)$, and write down p(x).
 - **3.2** Determine whether the resulting p is unique. Justify your answer.
 - **3.3** Show that one has the error formula

$$f(x) - p(x) = \frac{(x^2 - 1)^2 x}{5!} f^{(v)}(\xi)$$

for some $\xi(x) \in [-1, 1]$.

- Q4 4.1 Compute a quartic least-square approximation $p_4(x)$ to $f(x) = \cos(\pi x)$ for $x \in [-1, 1]$. Evaluate all integrals, but leave the final linear system unsolved.
 - **4.2** Continuing from **4.1**, show that, for any $n \in \mathbb{N}$, one can obtain a least-square approximation $p_n \in \mathcal{P}_n$ by showing that the linear system has a positive-definite matrix and is thus uniquely solvable.
- Q5 5.1 Show that the degree of exactness of the quadrature formula

$$\mathcal{Q}_2[f] = \frac{b-a}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]$$

in [a, b] is exactly 3.

- **5.2** Show that, for n odd, the degree of exactness of the n-node closed Newton–Cotes formula is at least n (instead of n 1).
- **5.3** Compute a set of polynomials $\{\phi_0, \dots, \phi_3\}$ orthogonal in [0, 1].
- **5.4** Compute x_j and ρ_j that give the highest accuracy for the quadrature formula

$$\int_0^1 f(x) \, \mathrm{d}x \simeq \rho_1 f(x_1) + \rho_2 f(x_2) + \rho_3 f(x_3).$$

5.5 For which f is your formula exact?