Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2020	MATH2051-WE01

Title: Numerical Analysis II

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. Show your working and explain your reasoning.

Revision:	

ED01/2020

Q1 **1.1** Explain what it means for a floating-point number to have (i) a *finite precision* and (ii) a *finite range*.

1.2 A student tried to use bisection in Python to solve $f_1(x) = x - 1 = 0$ and $f_3(x) = x^3 - 3x^2 + 3x - 1 = 0$, both with initial interval $[-1, 2]$. With f_1 he obtained 1.0 (the exact answer), while with f_3 he obtained 1.0000038146972656. Explain why f_3 gives a worse result, and how it relates to **1.1**.

1.3 Seeking instead to find the roots of $f_1(x) = x$ and $f_3(x) = x^3$, should we expect similar results (in terms of accuracy of the answer)? Justify your answer briefly.

Q2 **2.1** Determine the orders of convergence of the following sequences (all converging to 0): $x_n = (n!)^{-1}$, $y_n = 2^{-3^n}$ and $z_n = 5^{-n^2}$. Justify your answers briefly.

2.2 Show that the iteration $x_{k+1} = \cos x_k$ is a contraction mapping in some interval $I \subset [\frac{1}{2}, 1]$, and give one such interval.

2.3 Show that the same iteration $x_{k+1} = \cos x_k$ is in fact convergent for all $x_0 \in \mathbb{R}$.

Q3 Given $f(x) = 7\cos^2(\pi x/2)$, we seek a $p \in \mathcal{P}_4$ such that $p(-1) = f(-1)$, $p'(-1) = f'(-1)$, $p(0) = f(0)$, $p(1) = f(1)$ and $p'(1) = f'(1)$.

3.1 Compute the divided difference table using the usual Newton interpolation construction, i.e. with $[a, a]f := \lim_{y\to a}[a, y]f = f'(a)$, and write down $p(x)$.

3.2 Determine whether the resulting p is unique. Justify your answer.

3.3 Show that one has the error formula

$$f(x) - p(x) = \frac{(x^2 - 1)^2 x}{5!} f^{(v)}(\xi)$$

for some $\xi(x) \in [-1, 1]$.

Q4 **4.1** Compute a quartic least-square approximation $p_4(x)$ to $f(x) = \cos(\pi x)$ for $x \in [-1, 1]$. Evaluate all integrals, but leave the final linear system unsolved.

4.2 Continuing from **4.1**, show that, for any $n \in \mathbb{N}$, one can obtain a least-square approximation $p_n \in \mathcal{P}_n$ by showing that the linear system has a positive-definite matrix and is thus uniquely solvable.

Q5 **5.1** Show that the degree of exactness of the quadrature formula

$$\mathcal{Q}_2[f] = \frac{b-a}{6}\Big[f(a) + 4f\Big(\frac{a+b}{2}\Big) + f(b)\Big]$$

in $[a, b]$ is *exactly* 3.

5.2 Show that, for n odd, the degree of exactness of the n-node closed Newton–Cotes formula is at least n (instead of $n - 1$).

5.3 Compute a set of polynomials $\{\phi_0, \cdots, \phi_3\}$ orthogonal in $[0, 1]$.

5.4 Compute x_j and ρ_j that give the highest accuracy for the quadrature formula

$$\int_0^1 f(x)\, \mathrm{d}x \simeq \rho_1 f(x_1) + \rho_2 f(x_2) + \rho_3 f(x_3).$$

5.5 For which f is your formula exact?