EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2020	MATH2071-WE01

Title:

Mathematical Physics II

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. Show your working and explain your reasoning.

Revision:	

Q1 Consider a system described by the Lagrangian

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) - \frac{1}{2}(q_1^2 + q_2^2)\,.$$

1.1 Write down the Hamiltonian H for this system.

1.2 Show that the Poisson bracket satisfies

$$\{q_i, q_j\} = \{p_i, p_j\} = 0$$

and

$$\{q_i, p_j\} = \begin{cases} 1 & \text{if } i = j\,, \\ 0 & \text{if } i \neq j\,. \end{cases}$$

1.3 Consider the function $Q = q_1 p_2 - q_2 p_1$ in phase space. Show that the Poisson bracket $\{Q, H\}$ vanishes. Why does this imply that Q is conserved?

Consider now a one-dimensional quantum mechanical system. At $t = 0$ it is prepared in a state described by the wave function

$$\psi(t=0, x) = C\left(\frac{1}{\sqrt{2}}\psi_{E=1}(x) + e^{i\alpha}\psi_{E=2}(x)\right)\,,$$

where the wave functions on the right-hand side are orthonormal energy eigenfunctions. Both C and α are real constants.

1.4 Determine the constant C.

1.5 An energy measurement is made. What are the possible outcomes, and what are the probabilities of those outcomes?

1.6 Subsequently, the position of the particle is measured. What do you know about the wave function of the system immediately after this measurement?

Q2 Assume that we have a field $u(x,t)$ with Lagrangian density $\mathcal{L}(u, u_x, u_t)$, where $u_x = \frac{\partial u}{\partial x}$ and $u_t = \frac{\partial u}{\partial t}$, and action

$$S = \int dx dt \, \mathcal{L}(u, u_x, u_t) \, .$$

2.1 Show, using the stationary action principle (that is, $\delta S = 0$ to first order in δu around a solution of the equations of motion), that $u(x,t)$ obeys the Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial u_x}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial u_t}\right) = 0 \, .$$

(You can assume that $\delta u = 0$ for the boundary terms in the integral computing the variation of the action.)

2.2 Consider the case that $\mathcal{L} = \frac{1}{2}u_t^2 - \frac{1}{2}u_x^2$. Write the Euler-Lagrange equation for this system, and D'Alembert's general solution to it.

2.3 Consider now the case that the Lagrangian density $\mathcal{L}(u, u_x, u_t, u_{xx})$ depends also on $u_{xx} = \frac{\partial^2 u}{\partial x^2}$. Using the stationary action principle, show that the Euler-Lagrange equation that governs the dynamics in this case (again assuming that $\delta u = \delta u_x = 0$ for the boundary terms in the integral computing the variation of the action) is

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial u_x}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial u_t}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial \mathcal{L}}{\partial u_{xx}}\right) = 0 \, .$$

2.4 We will consider the special case

$$\mathcal{L}(u, u_x, u_t, u_{xx}) = \frac{1}{2}u_x u_t + u_x^3 - \frac{1}{2}u_{xx}^2 \, .$$

Derive the equations of motion for this system.

2.5 Assume that a solution of the form $u(x,t) = f(x-t)$ exists, for some function $f(\xi)$ of one parameter. Which differential equation must $f(\xi)$ satisfy so that $f(x-t)$ is a solution of the equations of motion for this system?

Q3 The energy for a system with Lagrangian $L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$ is defined to be

$$E = \left(\sum_{i=1}^{n} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right) - L\,.$$

3.1 Show, using the Euler-Lagrange equations of motion, that

$$\frac{dE}{dt} = -\frac{\partial L}{\partial t}\,.$$

3.2 Compute the energy associated to the Lagrangian

$$L = \frac{1}{2} \left(\sum_{i=1}^{n} K_i(q_1, \ldots, q_n, t) \dot{q}_i^2 \right) - V(q_1, \ldots, q_n, t)$$

with K_i and V arbitrary functions.

3.3 We now restrict to the case that K_i and V depend only on time and the differences of positions

$$(d_1, d_2, \ldots, d_n) = (q_1 - q_2, q_2 - q_3, \ldots, q_n - q_1)\,.$$

In other words

$$K_i(q_1, \ldots, q_n, t) = G_i(d_1, \ldots, d_n, t) \qquad ; \qquad V(q_1, \ldots, q_n, t) = W(d_1, \ldots, d_n, t)$$

for some functions G_i and W. Identify a symmetry of the system, and write the Noether charge Q associated to it.

3.4 Verify explicitly, taking the time derivatives, that the charge Q you just found is constant along a solution of the Euler-Lagrange equations.

Q4 This question concerns a quantum mechanical particle in the potential

$$V(x) = \begin{cases} 0\,, & x \leq 0\,, \\ V_0\,, & 0 < x < L\,, \\ 0\,, & x \geq 0\,. \end{cases}$$

Consider the wave function given by

$$\psi(x) = \begin{cases} e^{ikx} + Re^{-ikx}\,, & x < 0\,, \\ A + Bx\,, & 0 < x < L\,, \\ Te^{ikx}\,, & x > L\,, \end{cases}$$

with $k = \sqrt{2mV_0/\hbar^2}$ and complex constants A, B, R and T.

4.1 Show that the wave function given above is an eigenfunction of the Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m} + V(x)$. Determine the energy eigenvalue of this wave function.

4.2 How will this wave function change when the energy is increased?

4.3 What are the boundary conditions on $\psi(x)$ at $x = 0$ and $x = L$?

4.4 Use the boundary conditions to determine $|R|^2$ and $|T|^2$ in terms of k and L.

4.5 Discuss the limits $kL \to 0$ and $kL \to \infty$ by interpreting the constants R and T.

Q5 The simple harmonic oscillator is given by the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 \,.$$

5.1 Show that

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right)$$

is an eigenfunction of the Hamiltonian with eigenvalue $\frac{1}{2}\hbar\omega$.

In order to generate other wave functions, we can use the following operators

$$\hat{a} = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{x} + i\hat{p})\,, \qquad \hat{a}^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{x} - i\hat{p})\,.$$

5.2 Show that $\psi_0(x)$ is annihilated by $\hat{a}$, that is, show that $\hat{a}\psi_0(x) = 0$.

5.3 Show with an explicit computation that

$$[\hat{a}, \hat{a}^\dagger] = 1\,.$$

5.4 Argue based on this result that it is therefore possible to generate an infinite series of new energy eigenfunctions by acting repeatedly with $\hat{a}^\dagger$ on the ground state wave function $\psi_0(x)$. What is the energy eigenvalue of those wave functions?

5.5 Now compute explicitly

$$\psi_1(x) = \hat{a}^\dagger\, \psi_0(x)\,,$$

and also write down $\psi_1(t, x)$.

Finally, consider a special wave function which satisfies

$$\hat{a}\,\chi(t, x) = \alpha e^{-i\omega t}\chi(t, x)\,.$$

where α is a real constant.

5.6 Show that the expectation value of the position operator in the state corresponding to $\chi(t, x)$ is given by

$$\langle\hat{x}\rangle = \sqrt{\frac{2\hbar\alpha^2}{m\omega}}\cos(\omega t)\,.$$