Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2020	MATH2581-WE01

Title: Algebra II

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. Show your working and explain your reasoning.

Revision:	

ED01/2020

Q1 **1.1** Let R be an integral domain and let $a, b \in R$ satisfying $a^3 = b^3$ and $a^5 = b^5$. Prove that $a = b$.

1.2 Let $R = \mathbb{Z}[\sqrt{-13}] := \{a + b\sqrt{-13} : a, b \in \mathbb{Z}\} \subset \mathbb{C}$ be an integral domain. Throughout, you may use that the map $N : R \to \mathbb{Z} : N(a+b\sqrt{-13}) = a^2+13b^2$ is multiplicative, i.e., $N(xy) = N(x)N(y)$ for any $x, y \in R$, and the fact that the only units in this ring are ± 1.

(i) Prove that 2 is an irreducible element of R but is not a prime.

(ii) Prove that $\gcd(14, 7 + 7\sqrt{-13})$ does not exist in R.

(iii) Prove that the quotient ring $R/(2)$ has four elements.

(iv) Is $R/(2)$ isomorphic to either $\mathbb{Z}/4$ or $\mathbb{Z}/2 \times \mathbb{Z}/2$ as a ring?

Q2 **2.1** Prove that given any $a, b \in \mathbb{Z}$, the polynomial $x^2+\bar{a}x+\bar{b}$ is reducible in $(\mathbb{Z}/5)[x]$ if and only if there is an integer $y \in \mathbb{Z}$ satisfying $y^2 \equiv a^2 + b \bmod 5$.

2.2 List all pairs $\bar{a}, \bar{b} \in \mathbb{Z}/5$ such that the polynomial $x^2 + \bar{a}x + \bar{b}$ is irreducible in $(\mathbb{Z}/5)[x]$.

2.3 Factor $x^4 + x^3 + x^2 - \bar{3}x + \bar{1}$ into irreducibles in $(\mathbb{Z}/5)[x]$.

Let $f(x) = x^4 + x^3 + x^2 - 3x + 1$ in $\mathbb{Q}[x]$.

2.4 Show that $f(x)$ has no roots in $\mathbb{Q}$.

2.5 Prove that $f(x)$ is irreducible in $\mathbb{Q}[x]$.

Q3 **3.1** Let $R = (\mathbb{Z}/2)[x]/(x^2 + x + \bar{1})$.

(i) Without a proof, list all the elements of R.

(ii) Give addition and multiplication tables for R.

3.2 Prove that there is no non-trivial homomorphism $\varphi : \mathbb{Z}[\sqrt{2}] \to \mathbb{Z}/5$, between the rings $\mathbb{Z}[\sqrt{2}]$ and $\mathbb{Z}/5$.

3.3 Let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 2 & 5 & 4 & 6 & 1 & 7 \end{pmatrix}, \quad \text{and} \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 5 & 7 & 4 & 6 & 3 \end{pmatrix}$$

be permutations in S_7. Write σ and τ as products of disjoint cycles and compute $\sigma\tau$. Also compute $\sigma\tau\sigma^{-1}$ and its order as an element of S_7.

3.4 Let $(3\,7) \in S_7$. Find $x, y, z, w \in \{1, 2, \ldots, 7\}$ such that

$$(3\,7) = (3\,1\,2\,7)(3\,1\,7\,2)(x\,y\,z\,w).$$

Moreover, show that a transposition $(a\,b) \in S_n$, $n \geq 2$ can never be written as a product of two cycles of length three.

Q4 **4.1** Let $C = \langle x \rangle$ be a cyclic group of order 48, written multiplicatively. Find the positive integers a such that there exists a surjective homomorphism $\varphi_a : \mathbb{Z}/48 \to C$ with

$$\varphi_a(\bar{1}) = x^a.$$

4.2 Find all generators and all subgroups of $(\mathbb{Z}/13)^\times$.

4.3 Find the centre $Z = \{g \in D_4 \mid gx = xg \ \forall x \in D_4\}$ of the dihedral group D_4.

4.4 Determine the order of every element in the quotient group D_4/Z.

Q5 **5.1** Determine (up to isomorphism) all abelian groups of order 360. Show the details of your work and justify every step of the solution.

5.2 Determine all the conjugacy classes in the dihedral group D_7. Show the details of your work and justify every step of the solution.

5.3 Let G be a finite group. Show that the function $f : G \to G$, $f(g) = g^{-1}$, defines a bijection from the set $X = \{g \in G \mid g^2 \neq 1\}$ to itself. Use this to prove that if G has an even number of elements, then there exists an element $x \in G$ of order 2. *You may not use Cauchy's theorem.*