Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2020	MATH2627-WE01

Title:
Geometric Topology II

Time (for guidance only):	2 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. Questions in Section B carry **ONE and a HALF times** as many marks as those in Section A. Please start each question on a new page. Please write your CIS username at the top of each page. Show your working and explain your reasoning.

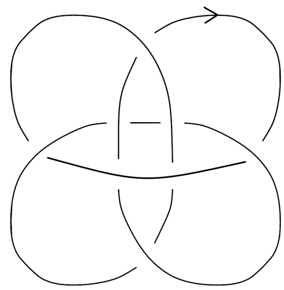
Revision:	

ED01/2020

SECTION A

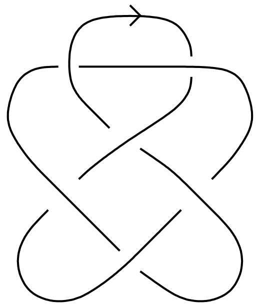
Q1 **1.1** Let U_n be the n-component unlink. State the values of $n \geq 1$ for which U_n is tricolorable

1.2 Denote by K_1 the following oriented knot.



Using tricolorability, show that there does not exist an oriented knot K_2 for which $K_1 + K_2 = U_1$.

1.3 Calculate the Alexander-Conway polynomial of the following knot. Do not assume knowledge of Alexander-Conway polynomials for links other than unlinks.



Q2 Consider the unit circle S^1 in $\mathbb{C}$.

2.1 Let $\overline{\gamma}(z) : S^1 \to S^1$ represent complex conjugation; that is $\overline{\gamma}(z) = \overline{z}$ for z in S^1. Show that the winding number $w(\overline{\gamma})$ of $\overline{\gamma}$ is -1.

2.2 Given any non-zero continuous map $f : S^1 \to \mathbb{C}$, recall that we may define a map $\gamma(z) : S^1 \to S^1$ by

$$\gamma(z) = \gamma_f(z) = \frac{f(z)}{|f(z)|}.$$

For the following functions f determine whether this γ is well-defined and, if so, compute the winding number $w(\gamma)$. You may find it useful to recall that for any two continuous functions $\gamma : S^1 \to S^1$ and $\delta : S^1 \to S^1$ we have

$$w(\gamma\delta) = w(\gamma) + w(\delta); \qquad w(\gamma \circ \delta) = w(\gamma)w(\delta).$$

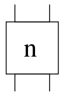
(i) $f(z) = \overline{z}(z^2 + \frac{i}{2})$;
(ii) $f(z) = z + \overline{z}$;
(iii) $f(z) = \overline{z}^3 - \overline{z}^2 + \frac{1}{2}\overline{z}$.

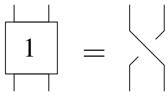
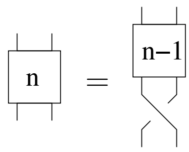
2.3 Let $\gamma_0, \gamma_1 : S^1 \to S^1$ be defined by

$$\gamma_0(z) = \frac{z^2 + \frac{1}{9}}{\left|z^2 + \frac{1}{9}\right|}; \qquad \gamma_1(z) = \frac{(1+i)z^2(\overline{z} - 2i)}{\sqrt{2}\ |\overline{z} - 2i|}.$$

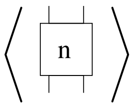
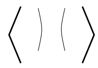
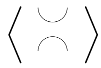
Find a continuous map $H : S^1 \times [0,1] \to S^1$ such that $H(z,0) = \gamma_0(z)$ and $H(z,1) = \gamma_1(z)$ for all z in S^1.

SECTION B

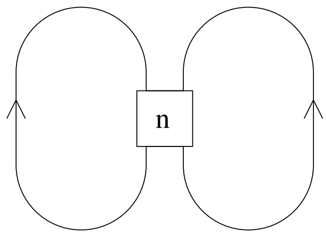
Q3 **3.1** For $n \geq 1$, let  be part of a link diagram defined as follows:

 and for $n > 1$ let  .

Use induction to show that for $n \geq 1$

 $= A^{-n}$  $+ \sum_{k=0}^{n-1} (-1)^{n-1-k} A^{3n-2-4k}$  .

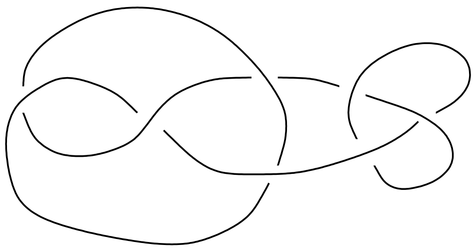
3.2 Denote by L_n the following link. Use part **3.1** to calculate the bracket polynomial of L_n (you need not group the terms).



3.3 Define the X-polynomial of an oriented link diagram in terms of the bracket polynomial, and calculate the X-polynomial of L_n.

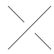
3.4 Calculate the Jones polynomial of L_3.

Q4 **4.1** Let K be the following knot.



(i) Use Seifert's algorithm on K to draw a Seifert surface.

(ii) Determine the genus of the resulting surface, and hence identify it.

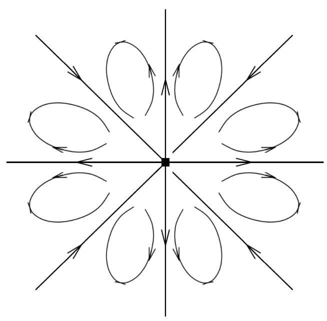
4.2 Let D be a knot diagram and let D' be a knot diagram obtained from D by changing a crossing of the form  to a crossing of the form .

(i) Briefly explain why applying Seifert's algorithm to both diagrams D and D' will give rise to surfaces of the same genus.

(ii) Show that changing a crossing on the standard diagram for the Trefoil leads to a diagram of the unknot. Hence, explain how one can draw a diagram of the unknot such that Seifert's algorithm produces a surface of genus 1 when applied to this diagram.

(iii) For every $n \geq 2$ give a diagram of the unknot such that Seifert's algorithm produces a surface of genus n when applied to this diagram.

Q5 **5.1** Determine the index of the following singularity. Justify your answer.



5.2 Draw a sketch of a planar vector field with exactly one singularity of index -3 and explain why it is indeed of index -3.

5.3 Let S be the 2-torus with one open disc removed. Draw a vector field on S which points outward along the boundary and which has only finitely many singularities. Calculate the sum of the indices of this vector field.

5.4 Suppose T is a compact connected orientable surface without boundary. Suppose also that there is a vector field on T with exactly three singularities, each of the same index. What are the possible values of the genus of T?

5.5 Does there exist a vector field on the surface of genus 3 with exactly four singularities such that two are of index 1 and two are of index -2? Justify your answer.