

EXAMINATION PAPER

Examination Session: May/June

2020

Year:

Exam Code:

MATH2647-WE01

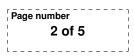
Title:

Probability II

| Time (for guidance only): | 2 hours | |
|-------------------------------|---------|---|
| Additional Material provided: | | |
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| Materials Permitted: | | |
| | | |
| Calculators Permitted: | Yes | Models Permitted: There is no restriction on the model of calculator which may be used. |

| Instructions to Candidates: | Credit will be given for your answers to all questions. Questions in Section B carry ONE and a HALF times as many marks as those in Section A. |
|-----------------------------|---|
| | Please start each question on a new page. Please write your CIS username at the top of each page. Show your working and explain your reasoning. |
| | |

Revision:





SECTION A

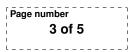
Q1 Let $(X_n)_{n\geq 0}$ be a (time-homogeneous) Markov chain with state space $S = \{1, 2, 3, 4, 5\}$ and transition matrix given by

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/3 & 2/3 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/3 & 1/6 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1/4 & 3/4 & 0 & 0 \end{pmatrix}.$$

- 1.1 Draw the directed graph associated with this Markov chain and find the communicating classes of this Markov chain.
- **1.2** Which of these communicating classes are closed? Are there any absorbing states?
- **1.3** Compute $\mathbb{P}[X_n = 1 \mid X_0 = 1]$ and $\mathbb{P}[X_n = 1 \mid X_0 = 2]$ for all $n \ge 1$.
- 1.4 Which states of this Markov chain are transient and which states are recurrent?

In your answer you should clearly state and carefully apply any result you use.

- Q2 2.1 State the Borel–Cantelli lemmas taking care to define any notation you introduce.
 - **2.2** A fair coin is tossed repeatedly. Let N denote the total number of heads observed. Assuming that the individual outcomes are independent, prove carefully that $\mathbb{P}(N = \infty) = 1$ in the following two ways:
 - (i) by using a suitable monotone approximation for the event $\{N = \infty\}$;
 - (ii) by using the Borel–Cantelli lemmas.



SECTION B

- **Q3** Let X be a random variable and let $(X_n)_{n>0}$ be a sequence of random variables.
 - **3.1** In the questions below, justify your answer by either proving the result or giving a counterexample.
 - (i) Does $\lim_{n\to\infty} X_n = X$ in L^1 imply that $\lim_{n\to\infty} X_n = X$ in probability?
 - (ii) Suppose that there exists $M \ge 0$ such that $\mathbb{P}[|X_n| < M] = 1$ for all $n \ge 1$ and that $\lim_{n\to\infty} X_n = 0$ in probability. Does $\lim_{n\to\infty} X_n = 0$ in L^1 ?
 - **3.2** Suppose that

$$\lim_{n \to \infty} \mathbb{E}\left[\frac{|X_n|}{1+|X_n|}\right] = 0.$$

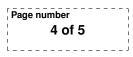
Show carefully that $\lim_{n\to\infty} X_n = 0$ in probability.

- **3.3** Suppose that $\lim_{n\to\infty} X_n = 0$ almost surely. By using question **3.2**, carefully prove that $\lim_{n\to\infty} X_n = 0$ in probability.
- **3.4** Distribute *n* different balls independently at random into *n* different boxes. Let N_n be the number of empty boxes, that is, set for $1 \le k \le n$

$$I_k = \begin{cases} 1 & \text{if } k^{\text{th}} \text{ box is empty} \\ 0 & \text{otherwise} \end{cases}$$

then $N_n = I_1 + \cdots + I_n$.

- (i) For $1 \leq i \leq n$, compute $\mathbb{E}[I_i]$ and $\mathbb{E}[N_n]$. For $1 \leq i, j \leq n$ with $i \neq j$, compute $\mathbb{E}[I_iI_j]$ and find $\mathbb{V}ar[N_n]$.
- (ii) Find $\lim_{n\to\infty} \mathbb{E}[N_n/n]$ and $\lim_{n\to\infty} \mathbb{V}ar[N_n/n]$. Show that N_n/n converges in probability and identify its limit.



Q4 Suppose that X_1, X_2, \ldots are independent random variables taking values 1 with probability p and -1 with probability q = 1 - p. Let $S_0 = 0$ and for $n \ge 1$ write $S_n = \sum_{i=1}^n X_i$. Let $p_n = \mathbb{P}[S_n = 0]$ for $n \ge 0$, $f_n = \mathbb{P}[S_k \ne 0$ for $1 \le k \le n - 1$ and $S_n = 0]$ for $n \ge 2$, and $f_1 = \mathbb{P}[S_1 = 0]$. Define

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$$P(s) = \sum_{n=0}^{\infty} p_n s^n$$
 and $F(s) = \sum_{n=1}^{\infty} f_n s^n$.

- **4.1** Show that S_n defines a Markov chain.
- 4.2 Justify the identity

$$p_n = \sum_{k=1}^n f_k p_{n-k}$$
, for $n \ge 1$.

- **4.3** Show that P(s) = 1 + P(s)F(s).
- **4.4** Compute explicitly p_n for $n \ge 0$ and deduce that $P(s) = (1 4pqs^2)^{-1/2}$. *Hint: you may use without proof the identity*

$$\sum_{n=0}^{\infty} {\binom{2n}{n}} s^n = (1-4s)^{-1/2}.$$

- **4.5** Find a formula for F(s) and compute F(1).
- **4.6** When is the Markov chain S_n recurrent and when is it transient?

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- **Q5** 5.1 A flea hops randomly on the vertices of a triangle, hopping to each of the other vertices with equal probability. Find the probability that after n hops the flea is back where it started.
 - **5.2** Let $(X_n)_{n\geq 0}$ be a Markov chain on $\{0, 1, 2, ...\}$ with transition probabilities given by

$$\mathbb{P}[X_n = 1 \mid X_{n-1} = 0] = 1, \qquad \mathbb{P}[X_n = k+1 \mid X_{n-1} = k] = p_{k,k+1}, \\ \mathbb{P}[X_n = k-1 \mid X_{n-1} = k] = p_{k,k-1}, \qquad \text{and } 0 \le p_{k,k+1} \le 1$$

for all $n, k \ge 1$. Furthermore, suppose that for all $k \ge 1$,

$$p_{k,k+1} = \left(\frac{k+1}{k}\right)^2 p_{k,k-1}$$

Let $H = \inf\{n \ge 0 : X_n = 0\}$ and define $h_i = \mathbb{P}[H < \infty \mid X_0 = i].$

(i) Write down a system of linear equations for $(h_k)_{k\geq 0}$ and verify that the solution of this system of linear equations is given by $h_0 = 1$ and for $k \geq 1$

$$h_k = 1 - A \sum_{r=1}^k \frac{1}{r^2},$$

where A is some constant.

(ii) Determine the constant A given above and find the probability that the chain never hits state 0 given that it started in state 2.*Hint: you may use without proof the identity*

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$