Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2020	MATH2647-WE01

Title:
Probability II

Time (for guidance only):	2 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. Questions in Section B carry **ONE and a HALF times** as many marks as those in Section A. Please start each question on a new page. Please write your CIS username at the top of each page. Show your working and explain your reasoning.

Revision:	

ED01/2020

SECTION A

Q1 Let $(X_n)_{n\geq 0}$ be a (time-homogeneous) Markov chain with state space $S = \{1, 2, 3, 4, 5\}$ and transition matrix given by

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/3 & 2/3 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/3 & 1/6 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1/4 & 3/4 & 0 & 0 \end{pmatrix}.$$

1.1 Draw the directed graph associated with this Markov chain and find the communicating classes of this Markov chain.

1.2 Which of these communicating classes are closed? Are there any absorbing states?

1.3 Compute $\mathbb{P}[X_n = 1 \mid X_0 = 1]$ and $\mathbb{P}[X_n = 1 \mid X_0 = 2]$ for all $n \geq 1$.

1.4 Which states of this Markov chain are transient and which states are recurrent?

In your answer you should clearly state and carefully apply any result you use.

Q2 **2.1** State the Borel–Cantelli lemmas taking care to define any notation you introduce.

2.2 A fair coin is tossed repeatedly. Let N denote the total number of heads observed. Assuming that the individual outcomes are independent, prove carefully that $\mathbb{P}(N = \infty) = 1$ in the following two ways:

(i) by using a suitable monotone approximation for the event $\{N = \infty\}$;

(ii) by using the Borel–Cantelli lemmas.

In your answer you should clearly state and carefully apply any result you use.

SECTION B

Q3 Let X be a random variable and let $(X_n)_{n\geq 0}$ be a sequence of random variables.

3.1 In the questions below, justify your answer by either proving the result or giving a counterexample.

(i) Does $\lim_{n\to\infty} X_n = X$ in L^1 imply that $\lim_{n\to\infty} X_n = X$ in probability?

(ii) Suppose that there exists $M \geq 0$ such that $\mathbb{P}[|X_n| < M] = 1$ for all $n \geq 1$ and that $\lim_{n\to\infty} X_n = 0$ in probability. Does $\lim_{n\to\infty} X_n = 0$ in L^1?

3.2 Suppose that

$$\lim_{n\to\infty} \mathbb{E}\left[\frac{|X_n|}{1+|X_n|}\right] = 0.$$

Show carefully that $\lim_{n\to\infty} X_n = 0$ in probability.

3.3 Suppose that $\lim_{n\to\infty} X_n = 0$ almost surely. By using question **3.2**, carefully prove that $\lim_{n\to\infty} X_n = 0$ in probability.

3.4 Distribute n different balls independently at random into n different boxes. Let N_n be the number of empty boxes, that is, set for $1 \leq k \leq n$

$$I_k = \begin{cases} 1 & \text{if } k^{\text{th}} \text{ box is empty} \\ 0 & \text{otherwise} \end{cases}$$

then $N_n = I_1 + \cdots + I_n$.

(i) For $1 \leq i \leq n$, compute $\mathbb{E}[I_i]$ and $\mathbb{E}[N_n]$. For $1 \leq i, j \leq n$ with $i \neq j$, compute $\mathbb{E}[I_i I_j]$ and find $\mathbb{V}\text{ar}[N_n]$.

(ii) Find $\lim_{n\to\infty} \mathbb{E}[N_n/n]$ and $\lim_{n\to\infty} \mathbb{V}\text{ar}[N_n/n]$. Show that N_n/n converges in probability and identify its limit.

In your answer you should clearly state and carefully apply any result you use.

Q4 Suppose that $X_1, X_2, \ldots$ are independent random variables taking values 1 with probability p and -1 with probability $q = 1 - p$. Let $S_0 = 0$ and for $n \geq 1$ write $S_n = \sum_{i=1}^{n} X_i$.

Let $p_n = \mathbb{P}[S_n = 0]$ for $n \geq 0$, $f_n = \mathbb{P}[S_k \neq 0 \text{ for } 1 \leq k \leq n-1 \text{ and } S_n = 0]$ for $n \geq 2$, and $f_1 = \mathbb{P}[S_1 = 0]$. Define

$$P(s) = \sum_{n=0}^{\infty} p_n s^n \quad \text{and} \quad F(s) = \sum_{n=1}^{\infty} f_n s^n.$$

4.1 Show that S_n defines a Markov chain.

4.2 Justify the identity

$$p_n = \sum_{k=1}^{n} f_k p_{n-k}, \text{ for } n \geq 1.$$

4.3 Show that $P(s) = 1 + P(s)F(s)$.

4.4 Compute explicitly p_n for $n \geq 0$ and deduce that $P(s) = (1 - 4pqs^2)^{-1/2}$.
Hint: you may use without proof the identity

$$\sum_{n=0}^{\infty} \binom{2n}{n} s^n = (1 - 4s)^{-1/2}.$$

4.5 Find a formula for $F(s)$ and compute $F(1)$.

4.6 When is the Markov chain S_n recurrent and when is it transient?

In your answer you should clearly state and carefully apply any result you use.

Q5 **5.1** A flea hops randomly on the vertices of a triangle, hopping to each of the other vertices with equal probability. Find the probability that after n hops the flea is back where it started.

5.2 Let $(X_n)_{n\geq 0}$ be a Markov chain on $\{0, 1, 2, \dots\}$ with transition probabilities given by

$$\mathbb{P}[X_n = 1 \mid X_{n-1} = 0] = 1, \qquad \mathbb{P}[X_n = k+1 \mid X_{n-1} = k] = p_{k,k+1},$$
$$\mathbb{P}[X_n = k-1 \mid X_{n-1} = k] = p_{k,k-1}, \qquad \text{and } 0 \leq p_{k,k+1} \leq 1$$

for all $n, k \geq 1$. Furthermore, suppose that for all $k \geq 1$,

$$p_{k,k+1} = \left(\frac{k+1}{k}\right)^2 p_{k,k-1}.$$

Let $H = \inf\{n \geq 0 : X_n = 0\}$ and define $h_i = \mathbb{P}[H < \infty \mid X_0 = i]$.

(i) Write down a system of linear equations for $(h_k)_{k\geq 0}$ and verify that the solution of this system of linear equations is given by $h_0 = 1$ and for $k \geq 1$

$$h_k = 1 - A \sum_{r=1}^{k} \frac{1}{r^2},$$

where A is some constant.

(ii) Determine the constant A given above and find the probability that the chain never hits state 0 given that it started in state 2.
Hint: you may use without proof the identity

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

In your answer you should clearly state and carefully apply any result you use.