Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2020	MATH2657-WE01

Title:

Special Relativity and Electromagnetism II

Time (for guidance only):	2 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. Questions in Section B carry **ONE and a HALF times** as many marks as those in Section A. Please start each question on a new page. Please write your CIS username at the top of each page. Show your working and explain your reasoning.

Revision:	

ED01/2020

SECTION A

Q1 [Take the speed of light to be $c = 3 \times 10^8\, ms^{-1}$].

1.1 Fry travels in a rocket ship towards Leela, at constant relative speed v. Fry is delivering a pizza, which in its rest frame stays hot for exactly another 2 minutes. If Leela measures that Fry is 27 million kilometers away, then calculate the minimal value of v for which the pizza is hot when delivered.

1.2 The microwave that was used to heat the pizza is on board the rocket and generates microwaves with a wavelength of $12\, cm$. Leela detects these microwaves but measures their wavelength to be $6\, cm$. Calculate the speed of the rocket.

Q2 **2.1** In an electrostatics problem, the electric field has the form

$$\mathbf{E} = \left(\frac{\alpha x y}{(\ell^2 + x^2)^2}, \frac{\beta}{\ell^2 + x^2}, 0 \right),$$

where α, β, ℓ are constants with $\ell > 0$.

Determine α in terms of β and find the electrostatic scalar potential ϕ and the electric charge density ρ, in terms of the spatial coordinates and the constants β, ℓ and possibly ε_0.

2.2 In a magnetostatics problem, the magnetic vector potential is

$$\mathbf{A} = (\lambda x^2 y, -\lambda y^2 x, 0),$$

where λ is a constant. Show that this vector potential is in Coulomb gauge and calculate the magnetic field $\mathbf{B}$ and the current density $\mathbf{J}$, in terms of the spatial coordinates and the constants λ and possibly μ_0.

SECTION B

Q3 **3.1** Let $\mathcal{R}$ be the rest frame of a skyscraper of proper height h. In this frame, the skyscraper has two identical circular racetracks, each of radius ρ. The first racetrack is on the ground floor and the second racetrack is on the roof. Seb drives around the ground floor racetrack at constant speed and Lewis drives around the roof racetrack, so that he is always directly above Seb. Both Seb and Lewis take a time T to complete each lap. Using spacetime coordinates t, x, y, z in the frame $\mathcal{R}$, the positions of Seb and Lewis are given by

$$(x_S, y_S, z_S) = (\rho\cos(2\pi t/T), \rho\sin(2\pi t/T), 0)$$

$$(x_L, y_L, z_L) = (\rho\cos(2\pi t/T), \rho\sin(2\pi t/T), h)\,.$$

Martin is travelling up the skyscraper in a lift moving with speed v. Martin's position is given by $(x_M, y_M, z_M) = (0, 0, vt)$.

Let $\mathcal{R}'$ be the frame with spacetime coordinates t', x', y', z' given by the Lorentz transformation, $\quad x' = x, \quad y' = y, \quad z' = \gamma(z - vt), \quad t' = \gamma(t - vz/c^2)$.

Calculate the positions of Martin, Seb and Lewis in the frame $\mathcal{R}'$ and hence answer the following questions, giving your answers in terms of h, T, c, γ.

(i) Show that $\mathcal{R}'$ is the rest frame of Martin.

(ii) Show that Martin measures both tracks to have radius ρ.

(iii) Calculate the height of the skyscraper as measured by Martin.

(iv) Show that Martin measures that both Seb and Lewis take the same time to complete a lap and calculate this lap time.

(v) Show that Martin observes that Lewis is not directly above Seb, except for some particular values of v which you should find an expression for.

(vi) Calculate the speed v for which Martin observes that Lewis is exactly half a lap ahead of Seb.

(vii) Derive a condition on T/h that makes it impossible for Martin to observe that Lewis is exactly half a lap ahead of Seb.

3.2 One of Maxwell's equations takes the form

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} + \lambda \frac{\partial \mathbf{E}}{\partial t},$$

where λ is a constant. Use Gauss' law and the continuity equation

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{J} = 0,$$

to determine λ in terms of the electric and magnetic constants.

Q4 **4.1** Use the integral form of Gauss' law to calculate the radial electric field due to the charge density $\rho = \alpha/(\ell^3 + r^3)^2$. Give your answer as a function of r and the positive constants $\alpha, \ell, \varepsilon_0$.

4.2 Let $\mathbf{m}$ be a constant vector. Show that, for $r \neq 0$, the vector potential

$$\mathbf{A} = \frac{\mathbf{m} \times \mathbf{r}}{r^3}$$

gives a magnetic field of the form

$$\mathbf{B} = \frac{(\mathbf{u} \cdot \widehat{\mathbf{r}})\widehat{\mathbf{r}} + \mathbf{v}}{r^3},$$

where $\mathbf{u}$ and $\mathbf{v}$ are constant vectors that you should determine in terms of $\mathbf{m}$.

4.3 Two particles, each of rest mass $2m$, collide and fuse to form a single third particle with rest mass $5m$. Find the speeds of the two initial particles in the rest frame of the final particle. Find the speed of one of the initial particles in the rest frame of the other initial particle. Give your answers as a fraction of the speed of light c.

Q5 **5.1** State the 4×4 matrix η for the Minkowski metric $\eta_{\mu\nu}$.

A Lorentz transformation is given by $x'^{\mu} = L^{\mu}{}_{\nu} x^{\nu}$.

Derive a condition, using index notation, on $L^{\mu}{}_{\nu}$, given that the interval $\eta_{\mu\nu} x^{\mu} x^{\nu}$ must be invariant under this transformation.

Write this condition as an equation in terms of η and the 4×4 matrix L, with entries $L^{\mu}{}_{\nu}$.

5.2 Find the value of a, given that the following matrix is a Lorentz transformation

$$L = \frac{1}{9} \begin{pmatrix} a & -12 & 0 & -20 \\ -20 & 15 & 0 & 16 \\ 0 & 0 & 9 & 0 \\ -12 & 0 & 0 & 15 \end{pmatrix}.$$

5.3 In terms of the electric and magnetic fields, $\mathbf{E} = (E_1, E_2, E_3)$ and $\mathbf{B} = (B_1, B_2, B_3)$, the electromagnetic field tensor is

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_1/c & -E_2/c & -E_3/c \\ E_1/c & 0 & -B_3 & B_2 \\ E_2/c & B_3 & 0 & -B_1 \\ E_3/c & -B_2 & B_1 & 0 \end{pmatrix},$$

where c is the speed of light.

Given that the magnetic field is $\mathbf{B} = (0, 0, b)$ and there is no electric field, calculate the electric and magnetic fields, $\mathbf{E}'$ and $\mathbf{B}'$ (in terms of b and c) on applying the Lorentz transformation that you found in the previous part of the question.

5.4 If space had four dimensions instead of three, write down the Minkowski metric η and the constraint equation satisfied by a Lorentz transformation matrix L. Use this constraint to count the number of parameters in a general Lorentz transformation in this theory with an extra spatial dimension.

5.5 If space had n dimensions, calculate the dimension of the Poincaré group.