EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2020	MATH2667-WE01

Title:
Monte Carlo II

Time (for guidance only):	1 hour 30 minutes	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. Show your working and explain your reasoning.

Revision:	

ED01/2020

Q1 **1.1** Write down an algorithm to sample the result of the flip of a fair coin.

1.2 Modify your algorithm to sample the result of the flip of a biased coin with probability p of observing heads.

1.3 In a simulation exercise, 100 coin flips were simulated from an algorithm with unknown p. A total of 36 heads were observed. Estimate the probability p of observing heads and calculate a 95% confidence interval. Comment on whether the algorithm was simulating a fair coin.

1.4 Say you have access to a fair six-sided die (or a program that simulates a fair six-sided die), you do not have access to any other resources capable of generating random numbers. Outline a method for sampling from a continuous Uniform distribution on the interval $(0, 1)$ using die rolls only. Comment on the precision of your method.

1.5 Say you have access to a fair coin (or a program that simulates a fair coin), you do not have access to any other resources capable of generating random numbers. Outline a method or algorithm for sampling from an Exponential distribution with mean 1.

Q2 Say we want to approximate a distribution by sampling using the crude Monte Carlo method and drawing a histogram of the results with K bins of equal width. We plan to take N samples from the Uniform distribution on the interval $(0, 1)$.

2.1 Write an estimate for the proportion of samples in each bin and a corresponding 95% confidence interval.

2.2 Assume we set our error tolerance at $x\%$. Find the required sample size N in terms of K and x.

2.3 Illustrate your results in questions **2.1** and **2.2** with $K = 20$ and $x = 1$ (1% error). Comment on the suitability of taking this approach when sampling from a distribution with unbounded support like the Normal distribution.

Q3 **3.1** Let X be a random variable with finite variance, and Y another random variable on the same probability space as X. Show that

$$\mathrm{Var}(X) = E\left(\mathrm{Var}(X|Y)\right) + \mathrm{Var}\left(E(X|Y)\right)$$

and conclude that $\mathrm{Var}(X) \geq \mathrm{Var}\left(E(X|Y)\right)$.

3.2 Let N be a random variable that represents the number of customers that visit the Palatine Cafe in a given day. Suppose $E(N)$ and $\mathrm{Var}(N)$ are known. Let X_i be the amount the ith, $i = 1, \ldots, N$, customer spends, and assume that the X_is are independent of each other and of N. Assume that each X_i follows an exponential distribution with mean 5. Calculate the expected value and variance of the Cafe's total sales on a given day.

3.3 After further investigation, it seems that the daily number of customers follows a Poisson distribution with mean $\lambda > 0$, that is $N \sim \mathrm{Poisson}(\lambda)$. We now would like to improve our estimate of total daily sales and possibly reduce our estimate's variance. Outline a strategy using stratified sampling or conditioning to improve your estimate of total sales and its corresponding variance.

Note that if $N \sim \mathrm{Poisson}(\lambda)$ then

$$P(N = n) = \frac{\lambda^n}{n!}\exp(-\lambda),\ n = 0, 1, \ldots.$$