

EXAMINATION PAPER

Examination Session: May/June

2020

Year:

Exam Code:

MATH3011-WE01

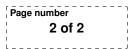
Title:

Analysis III

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks.
	Please start each question on a new page. Please write your CIS username at the top of each page.
	Show your working and explain your reasoning.

Revision:



- **Q1** 1.1 Let f be a nonnegative measurable function and suppose $\int f = 0$. Prove that f = 0 almost everywhere.
 - **1.2** Define what it means that $\phi : \mathbb{R} \to \mathbb{R}$ is simple and in standard representation, define $\int \phi$ for a simple function ϕ , and define $\int f$ for a measurable function $f : \mathbb{R} \to \mathbb{R}^{\geq 0}$.
 - **1.3** Prove that if $f, g : \mathbb{R} \to \mathbb{R}^{\geq 0}$ are measurable functions with $f(x) \leq g(x)$ for all $x \in \mathbb{R}$, then $\int f \leq \int g$.
 - 1.4 Let g be a bounded nonnegative measurable function. Show that for each $\epsilon > 0$ there is an integrable function f such that

$$\int fg \ge \left(\left\| g \right\|_{\infty} - \epsilon \right) \left\| f \right\|_{1}.$$

- Q2 2.1 State the Monotone Convergence Theorem. State the Lemma of Fatou.
 - **2.2** Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n := \chi_{[-2n,-n]}, n \in \mathbb{N}$. Here, $\chi_{[-2n,-n]}(x) = 1$ if $x \in [-2n, -n]$ and $\chi_{[-2n,-n]}(x) = 0$ otherwise. Does the assumption of the Monotone Convergence Theorem apply to the sequence f_n ? Does the conclusion of the Monotone Convergence Theorem apply to the sequence f_n ? Prove your answers.
 - **2.3** Let $g_n : \mathbb{R} \to \mathbb{R}$ be measurable, $\lim_{n \to \infty} g_n(x) = g(x)$ for all $x \in \mathbb{R}$ and assume that $0 \le g_n(x) \le g(x)$ for all $n \in \mathbb{N}$ and all $x \in \mathbb{R}$. Prove that $\lim_{n \to \infty} \int g_n = \int g$.
- **Q3 3.1** Define what it means for a measurable function $f : \mathbb{R} \to \mathbb{R}$ to be integrable and define $\int f$ for such f.
 - 3.2 State the Dominated Convergence Theorem.
 - **3.3** Use the identity $\int \frac{1}{1+x^2} = \pi$ and the Dominated Convergence Theorem to evaluate the limit $\lim_{n\to\infty} \int \frac{n\sin(x/n)}{x(1+x^2)}$.
- **Q4** 4.1 Define the Dirichlet kernel D_n and define the Fejer kernel F_n .
 - **4.2** Prove that, if $f : [-\pi, \pi] \to \mathbb{R}$ is integrable and if f is differentiable at $0 \in [-\pi, \pi]$, then

$$\lim_{n \to \infty} \int_{-\pi}^{\pi} f(y) D_n(y) dy = f(0).$$

- **Q5** 5.1 Define an inner product on $L^2([0, 2\pi])$ and define what it means that $L^2([0, 2\pi])$ is an Hilbert space.
 - 5.2 State Parseval's Identity.
 - **5.3** Let $\{e_i\}_{i \in I}$ be an orthonormal basis in a Hilbert space and let x be a unit vector, i.e., ||x|| = 1. Show that for each $k \in \mathbb{N}$ the set $\{i \in I : |\langle x, e_i \rangle| \ge 1/k\}$ has at most k^2 elements. [Hint: Use Parseval's Identity.]