Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2020	MATH3011-WE01

Title:
Analysis III

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. Show your working and explain your reasoning.

Revision:	

ED01/2020

Q1 **1.1** Let f be a nonnegative measurable function and suppose $\int f = 0$. Prove that $f = 0$ almost everywhere.

1.2 Define what it means that $\phi : \mathbb{R} \to \mathbb{R}$ is simple and in standard representation, define $\int \phi$ for a simple function ϕ, and define $\int f$ for a measurable function $f : \mathbb{R} \to \mathbb{R}^{\geq 0}$.

1.3 Prove that if $f, g : \mathbb{R} \to \mathbb{R}^{\geq 0}$ are measurable functions with $f(x) \leq g(x)$ for all $x \in \mathbb{R}$, then $\int f \leq \int g$.

1.4 Let g be a bounded nonnegative measurable function. Show that for each $\epsilon > 0$ there is an integrable function f such that

$$\int fg \geq (\|g\|_\infty - \epsilon) \|f\|_1 .$$

Q2 **2.1** State the Monotone Convergence Theorem. State the Lemma of Fatou.

2.2 Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n := \chi_{[-2n,-n]}, n \in \mathbb{N}$. Here, $\chi_{[-2n,-n]}(x) = 1$ if $x \in [-2n, -n]$ and $\chi_{[-2n,-n]}(x) = 0$ otherwise. Does the assumption of the Monotone Convergence Theorem apply to the sequence f_n? Does the conclusion of the Monotone Convergence Theorem apply to the sequence f_n? Prove your answers.

2.3 Let $g_n : \mathbb{R} \to \mathbb{R}$ be measurable, $\lim_{n\to\infty} g_n(x) = g(x)$ for all $x \in \mathbb{R}$ and assume that $0 \leq g_n(x) \leq g(x)$ for all $n \in \mathbb{N}$ and all $x \in \mathbb{R}$. Prove that $\lim_{n\to\infty} \int g_n = \int g$.

Q3 **3.1** Define what it means for a measurable function $f : \mathbb{R} \to \mathbb{R}$ to be integrable and define $\int f$ for such f.

3.2 State the Dominated Convergence Theorem.

3.3 Use the identity $\int \frac{1}{1+x^2} = \pi$ and the Dominated Convergence Theorem to evaluate the limit $\lim_{n\to\infty} \int \frac{n \sin(x/n)}{x(1+x^2)}$.

Q4 **4.1** Define the Dirichlet kernel D_n and define the Fejer kernel F_n.

4.2 Prove that, if $f : [-\pi, \pi] \to \mathbb{R}$ is integrable and if f is differentiable at $0 \in [-\pi, \pi]$, then

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} f(y) D_n(y) dy = f(0).$$

Q5 **5.1** Define an inner product on $L^2([0, 2\pi])$ and define what it means that $L^2([0, 2\pi])$ is an Hilbert space.

5.2 State Parseval's Identity.

5.3 Let $\{e_i\}_{i\in I}$ be an orthonormal basis in a Hilbert space and let x be a unit vector, i.e., $\|x\| = 1$. Show that for each $k \in \mathbb{N}$ the set $\{i \in I : |< x, e_i >| \geq 1/k\}$ has at most k^2 elements. [Hint: Use Parseval's Identity.]