EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2020	MATH3021-WE01

Title: Differential Geometry III

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. Show your working and explain your reasoning.

Revision:	

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Q1 Let $\boldsymbol{\alpha} : \mathbb{R} \to \mathbb{R}^3$ be a curve defined by $\boldsymbol{\alpha} : u \to (u, \cos\varphi \sin u, -\sin\varphi \sin u)$, where $\varphi \in \mathbb{R}$ is a constant.

1.1 Compute the curvature and the torsion of $\boldsymbol{\alpha}$.

1.2 Determine for which values of the constant φ is the trace $\boldsymbol{\alpha}$ contained in

(i) a straight line in $\mathbb{R}^3$,

(ii) a plane in $\mathbb{R}^3$.

1.3 Determine for which constants $c \in \mathbb{R}$ the equation $x^4 + y^4 + z^4 = c$ defines a regular surface. State explicitly all statements you use in your proofs.

1.4 Let S be a surface parametrised by $\boldsymbol{x}(u, v) = (u^2, u, v^2)$. Find the coefficients of the first and second fundamental forms. Compute the principal curvatures of S.

Q2 Let $\boldsymbol{\alpha} : \mathbb{R} \to \mathbb{R}^2$ be a curve given by $\boldsymbol{\alpha}(u) = (u^2 + u, u^3)$.

2.1 Is $\boldsymbol{\alpha}$ smooth? regular? simple? unit speed? Justify your answers.

2.2 Find all inflection points of $\boldsymbol{\alpha}$. Does $\boldsymbol{\alpha}$ have any vertices?

2.3 Find the evolute $\boldsymbol{e}(u)$ for the curve $\boldsymbol{\alpha}(u)$. For which $u \in \mathbb{R}$ is it defined?

2.4 Let $\boldsymbol{\gamma}$ be a curve satisfying all conditions of the 4-vertex theorem. Can $\boldsymbol{\gamma}$ have an odd number of vertices? Sketch a curve satisfying all conditions of the 4-vertex theorem and having exactly 12 vertices. (You do not need to write a formula).

Q3 Define the first fundamental form on the upper half-plane $U = \{(u, v) \in \mathbb{R}^2 \mid v > 0\}$ by

$$E(u, v) := \frac{1}{v^2}, \qquad F(u, v) := 0, \qquad G(u, v) := \frac{1}{v^2}.$$

3.1 Let $\boldsymbol{\alpha}(x) = (0, t)$, $1 \leq t < \infty$. Determine whether the arc length of $\boldsymbol{\alpha}$ is finite of infinite.

3.2 Find the area of the domain $T = \{(u, v) \in U \mid -1 \leq u \leq 1, u^2 + v^2 > 1\}$.

3.3 Give the definition of a global isometry of two surfaces. Show that the maps

$$f_a(u, v) = (au, av) \quad \text{and} \quad g_b(u, v) = (u + b, v)$$

are global isometries $U \to U$ for every $a, b \in \mathbb{R}$.

3.4 Consider the set C of curves which consists of all vertical half-rays (a, t), $a \in \mathbb{R}$, $t > 0$ in U and all semi-circles $(r\cos\theta + e, r\sin\theta)$, where $0 \leq \theta \leq \pi$ and $e \in \mathbb{R}$, $r \in \mathbb{R}_+$. Show without using the Gauss-Bonnet Theorem that every triangle on U with all three vertices on the boundary $\partial U \cup \infty$ and all sides contained in the set C has the same area π.

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Q4 Let $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = (\sin z + 2)^2\}$ be a surface.

4.1 Parametrise S as a surface of revolution. Write down the generating curve of S.

4.2 Compute the Gauss curvature of S. Find elliptic, parabolic and flat regions on S.

4.3 Is there any closed geodesic on S? Justify your answer.

4.4 Give the definition of an umbilic point. Are there any umbilic points on S? Justify your answer.

Q5 The surface $S \subset R^3$ is given by

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2, y \geq 0, 0 \leq z \leq 1\}.$$

5.1 State the global Gauss-Bonnet Theorem explaining all notions which you use.

5.2 Find the value of $\int_{\partial S} \kappa_g ds$.

5.3 Find the Euler characteristic of S.

5.4 Verify the global Gauss-Bonnet Theorem directly for the surface S.

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