Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2020	MATH3031-WE01

Title:
Number Theory III

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. Show your working and explain your reasoning.

Revision:	

ED01/2020

Q1 **1.1** In a similar way as for Pythagorean triples, find a formula giving exactly those primitive triples $(X, Y, Z) \in \left(\mathbb{Z}_{>0}\right)^3$ such that Y is even and

$$X^2 + 5Y^2 = Z^2 \,.$$

[Hints:

- You may want to determine the parity of X and of Z under the given assumptions.

- You may want to determine possible common divisors of $X+Z$ and $X-Z$.

- Your formula should depend on two parameters. You will also need to give appropriate parity and divisibility conditions.]

1.2 Show that $19+4\sqrt{-5}$ may be expressed as a product of two irreducible elements of $\mathbb{Z}[\sqrt{-5}]$.

1.3 Show that there is no solution in integers other than $(x, y, z) = (0, 0, 0)$ to

$$2x^{11} + 3y^{11} = 6z^{11} \,.$$

Q2 **2.1** Find the fundamental unit of $\mathbb{Z}[\sqrt{41}]$. Find formulas for all the solutions in integers, if any, to

$$x^2 - 41y^2 = -1 \,.$$

2.2 How many principal ideals in $\mathbb{Z}[\sqrt{-13}]$ are there of norm 119?

2.3 Show that for an integer $m \equiv 2 \pmod 3$, the number

$$\frac{m - \sqrt[3]{2}}{\sqrt[3]{3}}$$

is an algebraic integer.

2.4 Let $K = \mathbb{Q}(\theta)$ where θ is a root of $f(X) = X^4 + 2X^2 - 1$.
Find the degree $n = [K : \mathbb{Q}]$ and the discriminant $\Delta_K(\{1, \theta, \ldots, \theta^{n-1}\})$ and write down a basis for K over $\mathbb{Q}$.
Compute the norm $\mathrm{N}_K(2 - \theta^2)$ and the trace $\mathrm{Tr}_K(1 - 5\theta - 3\theta^2)$.

Q3 **3.1** Define the term *Euclidean function* for an integral domain S.

3.2 Show that $\mathcal{O}_{-11}$ possesses a Euclidean function.

3.3 Show that 2 is irreducible in $\mathcal{O}_{-11}$.

3.4 Show that for $\alpha \in \mathcal{O}_{-11}$ such that $\alpha\widetilde{\alpha} = 4N$ for some $N \in \mathbb{Z}_{>0}$, we have $\alpha \in \mathbb{Z}[\sqrt{-11}]$.

3.5 Give, in terms of a, b, $c \in \mathbb{Z}_{>0}$, a formula for the number of solutions (X, Y) in positive integers X, Y to the equation

$$X^2 + 11\,Y^2 = 2^2\,3^a\,5^b\,19^c$$

in the case where a, b, c are even.

Q4 Justifying carefully your method, find all the solutions $(X, Y) \in \mathbb{Z}^2$ to

$$X^2 + 35 = Y^3\,.$$

[You may assume that the class number of $\mathbb{Q}(\sqrt{-35})$ is 2.]

Q5 Determine the size and structure of the ideal class group of $\mathbb{Q}(\sqrt{-53})$. (Justify your answer.)

[You may use the Minkowski bound, given by $B_K = \left(\frac{4}{\pi}\right)^t \frac{n!}{n^n}\sqrt{|\Delta_K|}$ in the usual notation, but you should give the definition of the invariants t, n and Δ_K used in this formula.]