Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2020	MATH3041-WE01

Title:
Galois Theory III

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. Show your working and explain your reasoning.

Revision:	

Q1 **1.1** (i) Let $K = \mathbb{Q}(\sqrt[6]{2})$. Prove that K is not normal over $\mathbb{Q}$.

(ii) Find the minimal field extension L of K such that L is normal over $\mathbb{Q}$. (Justify your answer.)

1.2 For which prime numbers p is the field $\mathbb{F}_p(X)$ of rational functions in the variable X with coefficients in $\mathbb{F}_p$ separable over its subfield $\mathbb{F}_p(X^6)$?

1.3 Is the polynomial $X^6 + X^3 + 1 \in \mathbb{F}_2[X]$ irreducible? (Justify your answer.)

Q2 Suppose L is a splitting field for $X^4 - 3 \in \mathbb{Q}[X]$.

2.1 Prove that $i \in L$ and find a $\mathbb{Q}$-basis of L.

2.2 Describe the structure of the group $G = Gal(L/\mathbb{Q})$ by specifying its generators and relations.

2.3 Let $M = L(\sqrt[4]{2})$. Prove that M is Galois over $\mathbb{Q}$ and find $Gal(M/\mathbb{Q}(i))$.

2.4 Find all subfields $E \subset \mathbb{Q}(\sqrt[4]{2}, \sqrt[4]{3})$ such that $[E : \mathbb{Q}] = 4$.

Q3 **3.1** Find all complex roots of the polynomial $X^4 - X^2 + i\sqrt{6}X + 3/2$.

3.2 Suppose $\zeta \in \mathbb{C}$ is a 7-th primitive root of unity, $\alpha = \zeta + \zeta^2 + \zeta^4$ and $\beta = \zeta + \zeta^3$.

(i) Find $[\mathbb{Q}(\alpha) : \mathbb{Q}]$ and $[\mathbb{Q}(\beta) : \mathbb{Q}]$.

(ii) Find the minimal polynomial for α over $\mathbb{Q}$ and deduce that $\sqrt{-7} \in \mathbb{Q}(\zeta)$.

(iii) Find the minimal polynomial $P(T)$ for ζ over $\mathbb{Q}(\sqrt{-7})$. Prove that the discriminant of $P(T)$ equals A^2, where $A \in \mathbb{Q}(\sqrt{-7})$. Find this element A.

Q4 **4.1** Find the Galois groups of the following polynomials from $\mathbb{Q}[X]$:

(i) $X^4 - 2X + 2$;

(ii) $X^4 + 40X^2 + 80$.

4.2 Suppose $L = \mathbb{C}(X)$ and $K = \mathbb{C}(X^4 + X^{-4})$.

(i) Prove that the extension L/K is Galois and describe explicitly its Galois group (by specifying the corresponding generators and relations).

(ii) Find all subgroups H of order 4 in $Gal(L/K)$ and the corresponding subfields L^H.

Q5 **5.1** Prove that the polynomial $X^{12} + 2 \in \mathbb{F}_{13}[X]$ is irreducible.

5.2 How many irreducible monic polynomials of degree 12 are there in $\mathbb{F}_{13}[X]$?

5.3 (i) Give a construction of the field $\mathbb{F}_{16}$. Find all the roots of the polynomial $T^5 - 1$ in $\mathbb{F}_{16}$.

(ii) Find an irreducible polynomial of degree 5 in $\mathbb{F}_{16}[X]$.