

EXAMINATION PAPER

Examination Session: May/June

2020

Year:

Exam Code:

MATH3041-WE01

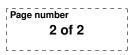
Title:

Galois Theory III

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks.
	Please start each question on a new page. Please write your CIS username at the top of each page.
	Show your working and explain your reasoning.

Revision:





Q1 1.1 (i) Let $K = \mathbb{Q}(\sqrt[6]{2})$. Prove that K is not normal over \mathbb{Q} .

- (ii) Find the minimal field extension L of K such that L is normal over \mathbb{Q} . (Justify your answer.)
- **1.2** For which prime numbers p is the field $\mathbb{F}_p(X)$ of rational functions in the variable X with coefficients in \mathbb{F}_p separable over its subfield $\mathbb{F}_p(X^6)$?
- **1.3** Is the polynomial $X^6 + X^3 + 1 \in \mathbb{F}_2[X]$ irreducible? (Justify your answer.)
- **Q2** Suppose L is a splitting field for $X^4 3 \in \mathbb{Q}[X]$.
 - **2.1** Prove that $i \in L$ and find a \mathbb{Q} -basis of L.
 - **2.2** Describe the structure of the group $G = Gal(L/\mathbb{Q})$ by specifying its generators and relations.
 - **2.3** Let $M = L(\sqrt[4]{2})$. Prove that M is Galois over \mathbb{Q} and find $Gal(M/\mathbb{Q}(i))$.
 - **2.4** Find all subfields $E \subset \mathbb{Q}(\sqrt[4]{2}, \sqrt[4]{3})$ such that $[E : \mathbb{Q}] = 4$.
- **Q3** 3.1 Find all complex roots of the polynomial $X^4 X^2 + i\sqrt{6}X + 3/2$.
 - **3.2** Suppose $\zeta \in \mathbb{C}$ is a 7-th primitive root of unity, $\alpha = \zeta + \zeta^2 + \zeta^4$ and $\beta = \zeta + \zeta^3$.
 - (i) Find $[\mathbb{Q}(\alpha) : \mathbb{Q}]$ and $[\mathbb{Q}(\beta) : \mathbb{Q}]$.
 - (ii) Find the minimal polynomial for α over \mathbb{Q} and deduce that $\sqrt{-7} \in \mathbb{Q}(\zeta)$.
 - (iii) Find the minimal polynomial P(T) for ζ over $\mathbb{Q}(\sqrt{-7})$. Prove that the discriminant of P(T) equals A^2 , where $A \in \mathbb{Q}(\sqrt{-7})$. Find this element A.
- **Q4** 4.1 Find the Galois groups of the following polynomials from $\mathbb{Q}[X]$:
 - (i) $X^4 2X + 2;$
 - (ii) $X^4 + 40X^2 + 80$.
 - **4.2** Suppose $L = \mathbb{C}(X)$ and $K = \mathbb{C}(X^4 + X^{-4})$.
 - (i) Prove that the extension L/K is Galois and describe explicitly its Galois group (by specifying the corresponding generators and relations).
 - (ii) Find all subgroups H of order 4 in Gal(L/K) and the corresponding subfields L^{H} .
- **Q5** 5.1 Prove that the polynomial $X^{12} + 2 \in \mathbb{F}_{13}[X]$ is irreducible.
 - **5.2** How many irreducible monic polynomials of degree 12 are there in $\mathbb{F}_{13}[X]$?
 - **5.3** (i) Give a construction of the field \mathbb{F}_{16} . Find all the roots of the polynomial $T^5 1$ in \mathbb{F}_{16} .
 - (ii) Find an irreducible polynomial of degree 5 in $\mathbb{F}_{16}[X]$.