EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2020	MATH3051-WE01

Title: Statistical Methods III

Time (for guidance only):	2 hours 30 minutes	
Additional Material provided:	Tables: Normal, t-distribution, F-distribution, χ^2-distribution; Graph paper.	
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. Show your working and explain your reasoning.

Revision:	

ED01/2020

Q1 **1.1** Let $\boldsymbol{A}$ and $\boldsymbol{B}$ denote symmetric $p \times p$ matrices. One says that a given square matrix is a *valid* variance matrix if it is symmetric and positive semi-definite.

(i) Show that

- if $\boldsymbol{A}$ possesses at least one negative value on the diagonal, then it cannot be valid;
- if $\boldsymbol{A}$ is a diagonal matrix with no negative values on the diagonal, then it is valid.

(ii) Prove or give counterexamples for the following two statements:

- If $\boldsymbol{A}$ and $\boldsymbol{B}$ are both valid variance matrices, then $\boldsymbol{A}+\boldsymbol{B}$ is a valid variance matrix.
- If $\boldsymbol{A}$ and $\boldsymbol{B}$ are both invalid variance matrices, then $\boldsymbol{A}+\boldsymbol{B}$ is an invalid variance matrix.

1.2 For a given random vector $X \sim N_q(\boldsymbol{m}, \boldsymbol{\Sigma})$, the squared *Mahalanobis distance to the mean* is given by

$$d_M^2(X, \boldsymbol{m}, \boldsymbol{\Sigma}) = (X - \boldsymbol{m})^T \boldsymbol{\Sigma}^{-1} (X - \boldsymbol{m}).$$

(i) Show that

$$d_M^2(X, \boldsymbol{m}, \boldsymbol{\Sigma}) \sim \chi_q^2\,, \tag{1}$$

where χ_q^2 denotes a χ^2 distribution with q degrees of freedom.

(ii) For a certain data set with $n = 48$ observations on $q = 3$ variables, the values of $d_M^2(\boldsymbol{x}_i, \hat{\boldsymbol{m}}, \hat{\boldsymbol{\Sigma}})$, where $\hat{\boldsymbol{m}}$ and $\hat{\boldsymbol{\Sigma}}$ are the sample mean and variance, respectively, are given by:

i	1	2	3	4	5	6	7	8
d_M^2	0.888	2.199	1.119	1.748	2.431	4.863	1.725	1.726
i	9	10	11	12	13	14	15	16
d_M^2	4.518	2.098	1.562	4.291	0.677	2.543	1.266	0.855
i	17	18	19	20	21	22	23	24
d_M^2	2.043	0.498	1.337	1.548	0.417	3.383	0.964	3.434
i	25	26	27	28	29	30	31	32
d_M^2	0.829	4.089	4.301	1.245	1.180	8.089	2.919	2.073
i	33	34	35	36	37	38	39	40
d_M^2	1.051	8.790	9.569	0.931	3.070	1.428	9.498	2.634
i	41	42	43	44	45	46	47	48
d_M^2	0.113	0.216	0.527	0.903	1.542	0.016	22.870	4.982

For each of the cases $i = 1, \ldots, 48$, we wish to test the null hypothesis H_0: 'Case i is not an outlier' vs. H_1: 'Case i is an outlier'. Carry out a suitable test, based on property (1), at the 2.5% level of significance. Do this first without, then with, the use of a Bonferroni correction. Explain your working, and write down for both scenarios the case numbers of the detected outliers.

Q2 2.1 An experiment was carried out to determine what factors affect torque, i.e. the work (measured as force × distance) required to tighten a locknut. The two factors considered were

- the plating type (*plating*): heat treated (HT), cadmium and wax plated (C&W), phosphate and oil plated (P&O);
- the test medium (*test*): mandrel or bolt.

The data are tabulated below in form of a two-way layout:

	Plating Type		
Test medium	C&W	HT	P&O
bolt	20, 16, 17, 18, 15 16, 19, 14, 15, 24	26, 40, 28, 38, 38 30, 26, 38, 45, 38	25, 40, 30, 17, 16 45, 49, 33, 30, 20
mandrel	24, 18, 17, 17, 15 23, 14, 18, 12, 11	32, 22, 30, 35, 32 28, 27, 28, 30, 30	10, 13, 17, 16, 15 14, 11, 14, 15, 16

(i) In the context of this data, explain the terms *treatment* and *replicates*, and make clear what it means to speak of *complete* and *balanced* factorial design.

(ii) We fit a sequence of linear models and observe the residual sum of squares, RSS:

Variables included	RSS
1	5752.3
1+ plating	3461.7
1+ plating + test	2640.3
1+ plating + test + plating:test	1975.2

(Here, a "1" symbolizes the intercept term and *plating:test* symbolizes the interaction of *plating* and *test*). From this, construct a sequential Analysis of Variance table. For all three sources of variation, give the F-values and test the null hypothesis: "the source does not contribute to the variation in the response" at the 5% level of significance.
Hint: You can use $F_{0.05}(1, 54) = 4.02$ and $F_{0.05}(2, 54) = 3.17$.

(iii) What is the RSS of a model containing only the intercept and *test*? Explain your answer.

[Question 2 continues on the next page]

2.2 (i) Mallows' $C_{\mathcal{I}}$ for a "submodel" of a "full" linear model with p terms may be written

$$C_{\mathcal{I}} = (p - p_{\mathcal{D}}) + p_{\mathcal{D}}\,(F_{\mathcal{D}} - 1)$$

where $\mathcal{I}$ is the subset of the indices $\{1, 2, \ldots, p\}$ of those terms included in the submodel, $\mathcal{D}$ is the subset of indices corresponding to those terms not included in the submodel, $p_{\mathcal{D}}$ is the number of excluded terms and $F_{\mathcal{D}}$ is the F-statistic for testing that the coefficients of the excluded terms are all zero.

Explain briefly how Mallows' $C_{\mathcal{I}}$ is used to select a submodel from a collection of submodels; and show that this is equivalent to choosing a model for which $F_{\mathcal{D}} \leq 1$ and $p_{\mathcal{D}}$ is as large as possible.

(ii) A study was carried out to investigate the variability of weights of machine components of standard size made by 4 different workers (W) on 2 days (D); and 5 components were chosen at random from the output of each worker on each day. The degrees-of-freedom (df) and sum-of-squares (SS) for each source of variation are given in the following ANOVA table:

Source	df	SS
Days (D)	1	540
Workers (W)	3	5470
Interaction (DW)	3	365
Residuals	32	3955
Total	39	10330

The table below shows selected entries of $F_{\mathcal{D}}$ for 5 submodels of the main effects-plus-interaction full model $1 + D + W + DW$.

- Fill in the missing values (marked ??) of $F_{\mathcal{D}}$.
- Use the result in part (b)(i) to find the best supported submodel according to Mallows' $C_{\mathcal{I}}$ criterion.

Submodel	$F_{\mathcal{D}}$
1	7.369
$1 + D$	??
$1 + W$	1.831
$1 + D + W$	??
$1 + D + W + DW$	1

Q3 Consider the linear regression model $y_i = \boldsymbol{d}_i^T \boldsymbol{\eta} + \epsilon_i$, $i = 1, \dots, n$, with parameters $\boldsymbol{\eta} = (\eta_1, \dots, \eta_p)^T$, and independent and normal errors $\epsilon_i \sim N(0, \sigma^2)$. Denote further by $\boldsymbol{D} = (\boldsymbol{d}_1^T, \dots, \boldsymbol{d}_n^T)^T$ the $n \times p$ design matrix, $\hat{\boldsymbol{\eta}}$ the least squares estimator of $\boldsymbol{\eta}$, and s^2 the usual unbiased estimator of σ^2. We know from lectures, and can use without proof, that the sampling distributions of $\hat{\boldsymbol{\eta}}$ and s^2 are given by $N_p(\boldsymbol{\eta}, \sigma^2(\boldsymbol{D}^T\boldsymbol{D})^{-1})$ and $\sigma^2\chi^2_{n-p}/(n-p)$, respectively.

3.1 Give an expression for the standard deviations, $\mathrm{SD}(\hat{\eta}_j)$, $j = 1, \dots, p$, and find the sampling distribution of

$$\frac{\hat{\eta}_j - \eta_j}{\mathrm{SD}(\hat{\eta}_j)}.$$

3.2 If one replaces, in the expression for $\mathrm{SD}(\hat{\eta}_j)$, the unknown value σ by its estimate, s, one refers to the resulting expression as the *standard error*, $\mathrm{SE}(\hat{\eta}_j)$, of $\hat{\eta}_j$. Show that the sampling distribution of the expression

$$T_j = \frac{\hat{\eta}_j - \eta_j}{\mathrm{SE}(\hat{\eta}_j)}, \tag{2}$$

for $j = 1, \dots, p$, is given by a t distribution with $n - p$ degrees of freedom. [*Note:* You can assume independence of s^2 and $\hat{\boldsymbol{\eta}}$ without proof.]

3.3 Explain how (2) can be used to devise a statistical test for the null hypothesis $H_0 : \eta_j = \eta_j^0$ versus $H_1 : \eta_j \neq \eta_j^0$, for some $\eta_j^0 \in \mathbb{R}$. Give the expressions for the test statistics as well the critical values explicitly.

3.4 Derive from (2) the expression for a $1-\alpha$ confidence interval for the 'true' value η_j. [*Note*: You do not need to have solved any of the previous subquestions to carry out this task.]

3.5 In an observational study, information on emotional wellbeing was collected from $n = 20$ pregnant women through questionnaires. The information was summarized in the form of three scores for anxiety (a), stress (s), and depression (d). We relate these scores through a linear model, considering anxiety as the response variable, i.e.,

$$a_i = \eta_1 + \eta_2 s_i + \eta_3 d_i + \epsilon_i, \quad i = 1, \dots, n.$$

Following least squares regression, 95% confidence intervals (CI) for η_2 and η_3 were obtained as follows:

j	CI for η_j
2	$(-0.092, 0.457)$
3	$(0.252, 1.857)$

(i) Interpret the two confidence intervals with reference to the term 'sampling distribution'.

(ii) Consider the four different null hypotheses $H_0 : \eta_j = k$, for $j = 2, 3$, $k = 0, 1$, each at the 5% level of significance. Carry out all four tests based on the given confidence intervals and interpret briefly the outcomes.

Q4 For the linear model $\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$, denote by $\hat{\boldsymbol{\beta}}$ the least squares estimator of $\boldsymbol{\beta}$. Let further $\hat{\boldsymbol{Y}} = \boldsymbol{X}\hat{\boldsymbol{\beta}}$ denote the vector of fitted values, $\hat{\boldsymbol{\epsilon}} = \boldsymbol{Y} - \hat{\boldsymbol{Y}}$ the vector of residuals, and $\text{RSS} = \hat{\boldsymbol{\epsilon}}^T\hat{\boldsymbol{\epsilon}}$ the residual sum of squares. Also, as usual, $\boldsymbol{H} = (h_{ij})_{1\le i,j\le n} = \boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T$ denotes the *hat matrix*.

4.1 Show that the vector of residuals can be written as $\hat{\boldsymbol{\epsilon}} = (\boldsymbol{I} - \boldsymbol{H})\boldsymbol{Y}$, with $\boldsymbol{I}$ denoting the identity matrix of appropriate dimension.

4.2 Find an expression for RSS which only depends on $\boldsymbol{H}$ and $\boldsymbol{Y}$, simplifying as far as possible.

4.3 Using parts **4.1** and **4.2**, or otherwise, show that

(i) $\hat{\boldsymbol{Y}}^T\hat{\boldsymbol{\epsilon}} = 0$

(ii) $\boldsymbol{Y}^T\hat{\boldsymbol{\epsilon}} = \hat{\boldsymbol{\epsilon}}^T\hat{\boldsymbol{\epsilon}}$

4.4 Show that $\text{Var}(\hat{\epsilon}_i) = (1-h_{ii})\sigma^2$ and $\text{Cov}(\hat{\epsilon}_i, \hat{y}_i) = 0$, and discuss briefly practical uses of these results.

4.5 Explain why

(i) $h_{ii} \le 1$;

(ii) h_{ii} is referred to as the *leverage* of the ith observation.

4.6 Cook's distance for the ith observation is given by

$$D_i = \frac{r_i^2}{p}\frac{h_{ii}}{1-h_{ii}}$$

where h_{ii} is leverage and r_i is the "internally studentised residual". Cook's distance is often used to measure influence. What is influence? How does it differ from leverage?

Q5 Assume $\mathbf{\Sigma} \in \mathbb{R}^{q\times q}$ is a valid variance matrix possessing only distinct eigenvalues. We consider a q-variate random vector $X = (X_1, \ldots, X_q)^T \sim (\mathbf{0}, \mathbf{\Sigma})$, where $q \geq 2$, and $\mathbf{0}$ is a vector of dimension q containing only zeros. Denote by $\lambda_1 > ... > \lambda_q$ the ordered eigenvalues of $\mathbf{\Sigma}$, and by $\boldsymbol{\gamma}_j = (\gamma_{j1} \ldots, \gamma_{jq})^T$ the corresponding j-th eigenvector, $j = 1, \ldots, q$. Let $\mathbf{\Gamma}$ denote the matrix whose columns contain the eigenvectors $\boldsymbol{\gamma}_j$ of $\mathbf{\Sigma}$ ordered by the size of their respective eigenvalues $\lambda_j, j = 1, \ldots, q$ (with the column corresponding to the largest eigenvalue coming first). Denote further

$$\mathbf{\Lambda}_r = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_r \end{pmatrix}$$

a diagonal matrix containing the first r ordered eigenvalues of $\mathbf{\Sigma}$, and by $\mathbf{\Gamma}_r$ a $q \times r$ matrix which has the corresponding eigenvector $\boldsymbol{\gamma}_j$ in its $j-$th column, $j = 1, \ldots, r$.

We are interested in approximating X by the best fitting bivariate linear subspace. We know that the solution to this problem is the plane through $\mathbf{0}$ spanned by the vectors $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$, with coordinates on this plane given by the principal component scores $\boldsymbol{\gamma}_1^T X$ and $\boldsymbol{\gamma}_2^T X$. We can combine these to form a bivariate random vector $T = \mathbf{\Gamma}_2^T X$.

5.1 Write down the eigen decomposition of $\mathbf{\Sigma}$, and use it to show that the sum of the eigenvalues equals the trace of $\mathbf{\Sigma}$.

5.2 The random vector T can be written as

$$T = \boldsymbol{v}_1 X_1 + \ldots \boldsymbol{v}_q X_q,$$

with appropriate vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_q \in \mathbb{R}^2$. Give expressions for the $\boldsymbol{v}_j, j = 1, \ldots, q$.

5.3 Show $\mathbf{\Sigma}\mathbf{\Gamma}_2 = \mathbf{\Gamma}_2\mathbf{\Lambda}_2$.

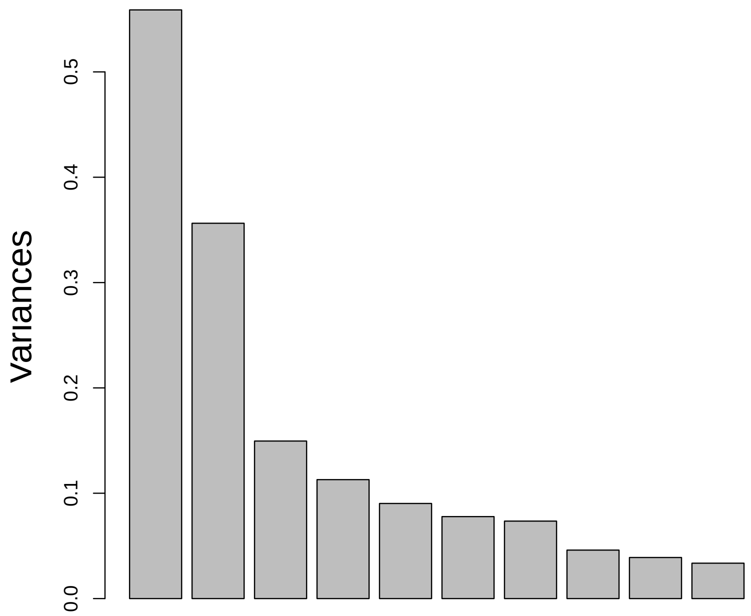
5.4 The bivariate coordinates T can be decompressed into the original (q–variate) data space via the operation $X' = \mathbf{\Gamma}_2 T$. Show that, if $q = 2$, then $X' = X$.

5.5 The ‘scree plot’ displayed on the following page was obtained through a principal component analysis of a data set of dimension $q = 60$, where each of the 60 variables measure the energy of sonar signals within a certain frequency band, after these signals bounced off from metallic objects or rocks. (The sonar signals were originally given on a scale from 0 to 1, but were mean–centered for this analysis in order to comply with the framework of this question).

(i) Explain what we see in this plot, referring to the notation outlined above.

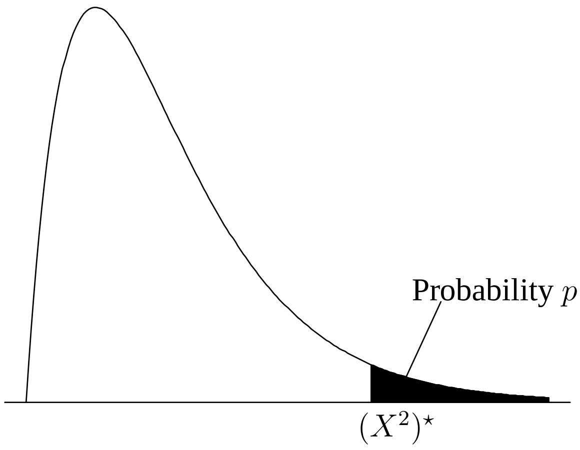
(ii) Would you deem it adequate to approximate this data set by a plane?

[Question 5 continues on the next page]


Variances
0.0
0.1
0.2
0.3
0.4
0.5

Probabilities for the χ^2-distribution

Table entry for p is the point $(X^2)^\star$ with probability p lying above it



	Tail probability p											
df	.995	.975	.25	.2	.1	.05	.025	.01	.005	.0025	.001	.0005
1	0.000039	0.00098	1.32	1.64	2.71	3.84	5.02	6.63	7.88	9.14	10.83	12.12
2	0.010	0.051	2.77	3.22	4.61	5.99	7.38	9.21	10.60	11.98	13.82	15.20
3	0.072	0.22	4.11	4.64	6.25	7.81	9.35	11.34	12.84	14.32	16.27	17.73
4	0.21	0.48	5.39	5.99	7.78	9.49	11.14	13.28	14.86	16.42	18.47	20.00
5	0.41	0.83	6.63	7.29	9.24	11.07	12.83	15.09	16.75	18.39	20.52	22.11
6	0.68	1.24	7.84	8.56	10.64	12.59	14.45	16.81	18.55	20.25	22.46	24.10
7	0.99	1.69	9.04	9.80	12.02	14.07	16.01	18.48	20.28	22.04	24.32	26.02
8	1.34	2.18	10.22	11.03	13.36	15.51	17.53	20.09	21.95	23.77	26.12	27.87
9	1.73	2.70	11.39	12.24	14.68	16.92	19.02	21.67	23.59	25.46	27.88	29.67
10	2.16	3.25	12.55	13.44	15.99	18.31	20.48	23.21	25.19	27.11	29.59	31.42
11	2.60	3.82	13.70	14.63	17.28	19.68	21.92	24.72	26.76	28.73	31.26	33.14
12	3.07	4.40	14.85	15.81	18.55	21.03	23.34	26.22	28.30	30.32	32.91	34.82
13	3.57	5.01	15.98	16.98	19.81	22.36	24.74	27.69	29.82	31.88	34.53	36.48
14	4.07	5.63	17.12	18.15	21.06	23.68	26.12	29.14	31.32	33.43	36.12	38.11
15	4.60	6.26	18.25	19.31	22.31	25.00	27.49	30.58	32.80	34.95	37.70	39.72
16	5.14	6.91	19.37	20.47	23.54	26.30	28.85	32.00	34.27	36.46	39.25	41.31
17	5.70	7.56	20.49	21.61	24.77	27.59	30.19	33.41	35.72	37.95	40.79	42.88
18	6.26	8.23	21.60	22.76	25.99	28.87	31.53	34.81	37.16	39.42	42.31	44.43
19	6.84	8.91	22.72	23.90	27.20	30.14	32.85	36.19	38.58	40.88	43.82	45.97
20	7.43	9.59	23.83	25.04	28.41	31.41	34.17	37.57	40.00	42.34	45.31	47.50
21	8.03	10.28	24.93	26.17	29.62	32.67	35.48	38.93	41.40	43.78	46.80	49.01
22	8.64	10.98	26.04	27.30	30.81	33.92	36.78	40.29	42.80	45.20	48.27	50.51
23	9.26	11.69	27.14	28.43	32.01	35.17	38.08	41.64	44.18	46.62	49.73	52.00
24	9.89	12.40	28.24	29.55	33.20	36.42	39.36	42.98	45.56	48.03	51.18	53.48
25	10.52	13.12	29.34	30.68	34.38	37.65	40.65	44.31	46.93	49.44	52.62	54.95
26	11.16	13.84	30.43	31.79	35.56	38.89	41.92	45.64	48.29	50.83	54.05	56.41
27	11.81	14.57	31.53	32.91	36.74	40.11	43.19	46.96	49.64	52.22	55.48	57.86
28	12.46	15.31	32.62	34.03	37.92	41.34	44.46	48.28	50.99	53.59	56.89	59.30
29	13.12	16.05	33.71	35.14	39.09	42.56	45.72	49.59	52.34	54.97	58.30	60.73
30	13.79	16.79	34.80	36.25	40.26	43.77	46.98	50.89	53.67	56.33	59.70	62.16
40	20.71	24.43	45.62	47.27	51.81	55.76	59.34	63.69	66.77	69.70	73.40	76.09
50	27.99	32.36	56.33	58.16	63.17	67.50	71.42	76.15	79.49	82.66	86.66	89.56
60	35.53	40.48	66.98	68.97	74.40	79.08	83.30	88.38	91.95	95.34	99.61	102.69
80	51.17	57.15	88.13	90.41	96.58	101.88	106.63	112.33	116.32	120.10	124.84	128.26
100	67.33	74.22	109.14	111.67	118.50	124.34	129.56	135.81	140.17	144.29	149.45	153.17

F distribution critical values

Denominator degrees of freedom	p	1	2	3	4	5	6	7	8	9
		Degrees of freedom in the numerator								
2	0.1	8.53	9.00	9.16	9.24	9.29	9.33	9.35	9.37	9.38
	0.05	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38
	0.025	38.51	39.00	39.17	39.25	39.30	39.33	39.36	39.37	39.39
	0.01	98.50	99.00	99.17	99.25	99.30	99.33	99.36	99.37	99.39
	0.001	998.50	999.00	999.17	999.25	999.30	999.33	999.36	999.37	999.39
3	0.1	5.54	5.46	5.39	5.34	5.31	5.28	5.27	5.25	5.24
	0.05	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81
	0.025	17.44	16.04	15.44	15.10	14.88	14.73	14.62	14.54	14.47
	0.01	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.35
	0.001	167.03	148.50	141.11	137.10	134.58	132.85	131.58	130.62	129.86
4	0.1	4.54	4.32	4.19	4.11	4.05	4.01	3.98	3.95	3.94
	0.05	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00
	0.025	12.22	10.65	9.98	9.60	9.36	9.20	9.07	8.98	8.90
	0.01	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66
	0.001	74.14	61.25	56.18	53.44	51.71	50.53	49.66	49.00	48.47
5	0.1	4.06	3.78	3.62	3.52	3.45	3.40	3.37	3.34	3.32
	0.05	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77
	0.025	10.01	8.43	7.76	7.39	7.15	6.98	6.85	6.76	6.68
	0.01	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16
	0.001	47.18	37.12	33.20	31.09	29.75	28.83	28.16	27.65	27.24
6	0.1	3.78	3.46	3.29	3.18	3.11	3.05	3.01	2.98	2.96
	0.05	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10
	0.025	8.81	7.26	6.60	6.23	5.99	5.82	5.70	5.60	5.52
	0.01	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.98
	0.001	35.51	27.00	23.70	21.92	20.80	20.03	19.46	19.03	18.69
7	0.1	3.59	3.26	3.07	2.96	2.88	2.83	2.78	2.75	2.72
	0.05	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68
	0.025	8.07	6.54	5.89	5.52	5.29	5.12	4.99	4.90	4.82
	0.01	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72
	0.001	29.25	21.69	18.77	17.20	16.21	15.52	15.02	14.63	14.33
8	0.1	3.46	3.11	2.92	2.81	2.73	2.67	2.62	2.59	2.56
	0.05	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39
	0.025	7.57	6.06	5.42	5.05	4.82	4.65	4.53	4.43	4.36
	0.01	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91
	0.001	25.41	18.49	15.83	14.39	13.48	12.86	12.40	12.05	11.77
9	0.1	3.36	3.01	2.81	2.69	2.61	2.55	2.51	2.47	2.44
	0.05	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18
	0.025	7.21	5.71	5.08	4.72	4.48	4.32	4.20	4.10	4.03
	0.01	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35
	0.001	22.86	16.39	13.90	12.56	11.71	11.13	10.70	10.37	10.11
10	0.1	3.29	2.92	2.73	2.61	2.52	2.46	2.41	2.38	2.35
	0.05	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02
	0.025	6.94	5.46	4.83	4.47	4.24	4.07	3.95	3.85	3.78
	0.01	10.04	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94
	0.001	21.04	14.91	12.55	11.28	10.48	9.93	9.52	9.20	8.96
11	0.1	3.23	2.86	2.66	2.54	2.45	2.39	2.34	2.30	2.27
	0.05	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90
	0.025	6.72	5.26	4.63	4.28	4.04	3.88	3.76	3.66	3.59
	0.01	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63
	0.001	19.69	13.81	11.56	10.35	9.58	9.05	8.66	8.35	8.12
12	0.1	3.18	2.81	2.61	2.48	2.39	2.33	2.28	2.24	2.21
	0.05	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80
	0.025	6.55	5.10	4.47	4.12	3.89	3.73	3.61	3.51	3.44
	0.01	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39
	0.001	18.64	12.97	10.80	9.63	8.89	8.38	8.00	7.71	7.48
13	0.1	3.14	2.76	2.56	2.43	2.35	2.28	2.23	2.20	2.16
	0.05	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71
	0.025	6.41	4.97	4.35	4.00	3.77	3.60	3.48	3.39	3.31
	0.01	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19
	0.001	17.82	12.31	10.21	9.07	8.35	7.86	7.49	7.21	6.98

F distribution critical values

Denominator degrees of freedom	p	1	2	3	4	5	6	7	8	9
		Degrees of freedom in the numerator								
14	0.1	3.10	2.73	2.52	2.39	2.31	2.24	2.19	2.15	2.12
	0.05	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65
	0.025	6.30	4.86	4.24	3.89	3.66	3.50	3.38	3.29	3.21
	0.01	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	4.03
	0.001	17.14	11.78	9.73	8.62	7.92	7.44	7.08	6.80	6.58
15	0.1	3.07	2.70	2.49	2.36	2.27	2.21	2.16	2.12	2.09
	0.05	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59
	0.025	6.20	4.77	4.15	3.80	3.58	3.41	3.29	3.20	3.12
	0.01	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89
	0.001	16.59	11.34	9.34	8.25	7.57	7.09	6.74	6.47	6.26
16	0.1	3.05	2.67	2.46	2.33	2.24	2.18	2.13	2.09	2.06
	0.05	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54
	0.025	6.12	4.69	4.08	3.73	3.50	3.34	3.22	3.12	3.05
	0.01	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78
	0.001	16.12	10.97	9.01	7.94	7.27	6.80	6.46	6.19	5.98
17	0.1	3.03	2.64	2.44	2.31	2.22	2.15	2.10	2.06	2.03
	0.05	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49
	0.025	6.04	4.62	4.01	3.66	3.44	3.28	3.16	3.06	2.98
	0.01	8.40	6.11	5.18	4.67	4.34	4.10	3.93	3.79	3.68
	0.001	15.72	10.66	8.73	7.68	7.02	6.56	6.22	5.96	5.75
18	0.1	3.01	2.62	2.42	2.29	2.20	2.13	2.08	2.04	2.00
	0.05	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46
	0.025	5.98	4.56	3.95	3.61	3.38	3.22	3.10	3.01	2.93
	0.01	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60
	0.001	15.38	10.39	8.49	7.46	6.81	6.35	6.02	5.76	5.56
19	0.1	2.99	2.61	2.40	2.27	2.18	2.11	2.06	2.02	1.98
	0.05	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42
	0.025	5.92	4.51	3.90	3.56	3.33	3.17	3.05	2.96	2.88
	0.01	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52
	0.001	15.08	10.16	8.28	7.27	6.62	6.18	5.85	5.59	5.39
20	0.1	2.97	2.59	2.38	2.25	2.16	2.09	2.04	2.00	1.96
	0.05	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39
	0.025	5.87	4.46	3.86	3.51	3.29	3.13	3.01	2.91	2.84
	0.01	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46
	0.001	14.82	9.95	8.10	7.10	6.46	6.02	5.69	5.44	5.24
21	0.1	2.96	2.57	2.36	2.23	2.14	2.08	2.02	1.98	1.95
	0.05	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37
	0.025	5.83	4.42	3.82	3.48	3.25	3.09	2.97	2.87	2.80
	0.01	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40
	0.001	14.59	9.77	7.94	6.95	6.32	5.88	5.56	5.31	5.11
22	0.1	2.95	2.56	2.35	2.22	2.13	2.06	2.01	1.97	1.93
	0.05	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34
	0.025	5.79	4.38	3.78	3.44	3.22	3.05	2.93	2.84	2.76
	0.01	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35
	0.001	14.38	9.61	7.80	6.81	6.19	5.76	5.44	5.19	4.99
23	0.1	2.94	2.55	2.34	2.21	2.11	2.05	1.99	1.95	1.92
	0.05	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32
	0.025	5.75	4.35	3.75	3.41	3.18	3.02	2.90	2.81	2.73
	0.01	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30
	0.001	14.20	9.47	7.67	6.70	6.08	5.65	5.33	5.09	4.89
24	0.1	2.93	2.54	2.33	2.19	2.10	2.04	1.98	1.94	1.91
	0.05	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30
	0.025	5.72	4.32	3.72	3.38	3.15	2.99	2.87	2.78	2.70
	0.01	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26
	0.001	14.03	9.34	7.55	6.59	5.98	5.55	5.23	4.99	4.80
25	0.1	2.92	2.53	2.32	2.18	2.09	2.02	1.97	1.93	1.89
	0.05	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28
	0.025	5.69	4.29	3.69	3.35	3.13	2.97	2.85	2.75	2.68
	0.01	7.77	5.57	4.68	4.18	3.85	3.63	3.46	3.32	3.22
	0.001	13.88	9.22	7.45	6.49	5.89	5.46	5.15	4.91	4.71

F distribution critical values

Denominator degrees of freedom	p	1	2	3	4	5	6	7	8	9
		Degrees of freedom in the numerator								
26	0.1	2.91	2.52	2.31	2.17	2.08	2.01	1.96	1.92	1.88
	0.05	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27
	0.025	5.66	4.27	3.67	3.33	3.10	2.94	2.82	2.73	2.65
	0.01	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18
	0.001	13.74	9.12	7.36	6.41	5.80	5.38	5.07	4.83	4.64
27	0.1	2.90	2.51	2.30	2.17	2.07	2.00	1.95	1.91	1.87
	0.05	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25
	0.025	5.63	4.24	3.65	3.31	3.08	2.92	2.80	2.71	2.63
	0.01	7.68	5.49	4.60	4.11	3.78	3.56	3.39	3.26	3.15
	0.001	13.61	9.02	7.27	6.33	5.73	5.31	5.00	4.76	4.57
28	0.1	2.89	2.50	2.29	2.16	2.06	2.00	1.94	1.90	1.87
	0.05	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24
	0.025	5.61	4.22	3.63	3.29	3.06	2.90	2.78	2.69	2.61
	0.01	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12
	0.001	13.50	8.93	7.19	6.25	5.66	5.24	4.93	4.69	4.50
29	0.1	2.89	2.50	2.28	2.15	2.06	1.99	1.93	1.89	1.86
	0.05	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22
	0.025	5.59	4.20	3.61	3.27	3.04	2.88	2.76	2.67	2.59
	0.01	7.60	5.42	4.54	4.04	3.73	3.50	3.33	3.20	3.09
	0.001	13.39	8.85	7.12	6.19	5.59	5.18	4.87	4.64	4.45
30	0.1	2.88	2.49	2.28	2.14	2.05	1.98	1.93	1.88	1.85
	0.05	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21
	0.025	5.57	4.18	3.59	3.25	3.03	2.87	2.75	2.65	2.57
	0.01	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07
	0.001	13.29	8.77	7.05	6.12	5.53	5.12	4.82	4.58	4.39
40	0.1	2.84	2.44	2.23	2.09	2.00	1.93	1.87	1.83	1.79
	0.05	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12
	0.025	5.42	4.05	3.46	3.13	2.90	2.74	2.62	2.53	2.45
	0.01	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89
	0.001	12.61	8.25	6.59	5.70	5.13	4.73	4.44	4.21	4.02
50	0.1	2.81	2.41	2.20	2.06	1.97	1.90	1.84	1.80	1.76
	0.05	4.03	3.18	2.79	2.56	2.40	2.29	2.20	2.13	2.07
	0.025	5.34	3.97	3.39	3.05	2.83	2.67	2.55	2.46	2.38
	0.01	7.17	5.06	4.20	3.72	3.41	3.19	3.02	2.89	2.78
	0.001	12.22	7.96	6.34	5.46	4.90	4.51	4.22	4.00	3.82
60	0.1	2.79	2.39	2.18	2.04	1.95	1.87	1.82	1.77	1.74
	0.05	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04
	0.025	5.29	3.93	3.34	3.01	2.79	2.63	2.51	2.41	2.33
	0.01	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72
	0.001	11.97	7.77	6.17	5.31	4.76	4.37	4.09	3.86	3.69
100	0.1	2.76	2.36	2.14	2.00	1.91	1.83	1.78	1.73	1.69
	0.05	3.94	3.09	2.70	2.46	2.31	2.19	2.10	2.03	1.97
	0.025	5.18	3.83	3.25	2.92	2.70	2.54	2.42	2.32	2.24
	0.01	6.90	4.82	3.98	3.51	3.21	2.99	2.82	2.69	2.59
	0.001	11.50	7.41	5.86	5.02	4.48	4.11	3.83	3.61	3.44
200	0.1	2.73	2.33	2.11	1.97	1.88	1.80	1.75	1.70	1.66
	0.05	3.89	3.04	2.65	2.42	2.26	2.14	2.06	1.98	1.93
	0.025	5.10	3.76	3.18	2.85	2.63	2.47	2.35	2.26	2.18
	0.01	6.76	4.71	3.88	3.41	3.11	2.89	2.73	2.60	2.50
	0.001	11.15	7.15	5.63	4.81	4.29	3.92	3.65	3.43	3.26
1000	0.1	2.71	2.31	2.09	1.95	1.85	1.78	1.72	1.68	1.64
	0.05	3.85	3.00	2.61	2.38	2.22	2.11	2.02	1.95	1.89
	0.025	5.04	3.70	3.13	2.80	2.58	2.42	2.30	2.20	2.13
	0.01	6.66	4.63	3.80	3.34	3.04	2.82	2.66	2.53	2.43
	0.001	10.89	6.96	5.46	4.65	4.14	3.78	3.51	3.30	3.13

F distribution critical values

Denominator degrees of freedom	p	10	11	12	13	14	15	16	17	18	19
		Degrees of freedom in the numerator									
2	0.1	9.39	9.40	9.41	9.41	9.42	9.42	9.43	9.43	9.44	9.44
	0.05	19.40	19.40	19.41	19.42	19.42	19.43	19.43	19.44	19.44	19.44
	0.025	39.40	39.41	39.41	39.42	39.43	39.43	39.44	39.44	39.44	39.45
	0.01	99.40	99.41	99.42	99.42	99.43	99.43	99.44	99.44	99.44	99.45
	0.001	999.40	999.41	999.42	999.42	999.43	999.43	999.44	999.44	999.44	999.45
3	0.1	5.23	5.22	5.22	5.21	5.20	5.20	5.20	5.19	5.19	5.19
	0.05	8.79	8.76	8.74	8.73	8.71	8.70	8.69	8.68	8.67	8.67
	0.025	14.42	14.37	14.34	14.30	14.28	14.25	14.23	14.21	14.20	14.18
	0.01	27.23	27.13	27.05	26.98	26.92	26.87	26.83	26.79	26.75	26.72
	0.001	129.25	128.74	128.32	127.96	127.64	127.37	127.14	126.93	126.74	126.57
4	0.1	3.92	3.91	3.90	3.89	3.88	3.87	3.86	3.86	3.85	3.85
	0.05	5.96	5.94	5.91	5.89	5.87	5.86	5.84	5.83	5.82	5.81
	0.025	8.84	8.79	8.75	8.71	8.68	8.66	8.63	8.61	8.59	8.58
	0.01	14.55	14.45	14.37	14.31	14.25	14.20	14.15	14.11	14.08	14.05
	0.001	48.05	47.70	47.41	47.16	46.95	46.76	46.60	46.45	46.32	46.21
5	0.1	3.30	3.28	3.27	3.26	3.25	3.24	3.23	3.22	3.22	3.21
	0.05	4.74	4.70	4.68	4.66	4.64	4.62	4.60	4.59	4.58	4.57
	0.025	6.62	6.57	6.52	6.49	6.46	6.43	6.40	6.38	6.36	6.34
	0.01	10.05	9.96	9.89	9.82	9.77	9.72	9.68	9.64	9.61	9.58
	0.001	26.92	26.65	26.42	26.22	26.06	25.91	25.78	25.67	25.57	25.48
6	0.1	2.94	2.92	2.90	2.89	2.88	2.87	2.86	2.85	2.85	2.84
	0.05	4.06	4.03	4.00	3.98	3.96	3.94	3.92	3.91	3.90	3.88
	0.025	5.46	5.41	5.37	5.33	5.30	5.27	5.24	5.22	5.20	5.18
	0.01	7.87	7.79	7.72	7.66	7.60	7.56	7.52	7.48	7.45	7.42
	0.001	18.41	18.18	17.99	17.82	17.68	17.56	17.45	17.35	17.27	17.19
7	0.1	2.70	2.68	2.67	2.65	2.64	2.63	2.62	2.61	2.61	2.60
	0.05	3.64	3.60	3.57	3.55	3.53	3.51	3.49	3.48	3.47	3.46
	0.025	4.76	4.71	4.67	4.63	4.60	4.57	4.54	4.52	4.50	4.48
	0.01	6.62	6.54	6.47	6.41	6.36	6.31	6.28	6.24	6.21	6.18
	0.001	14.08	13.88	13.71	13.56	13.43	13.32	13.23	13.14	13.06	12.99
8	0.1	2.54	2.52	2.50	2.49	2.48	2.46	2.45	2.45	2.44	2.43
	0.05	3.35	3.31	3.28	3.26	3.24	3.22	3.20	3.19	3.17	3.16
	0.025	4.30	4.24	4.20	4.16	4.13	4.10	4.08	4.05	4.03	4.02
	0.01	5.81	5.73	5.67	5.61	5.56	5.52	5.48	5.44	5.41	5.38
	0.001	11.54	11.35	11.19	11.06	10.94	10.84	10.75	10.67	10.60	10.54
9	0.1	2.42	2.40	2.38	2.36	2.35	2.34	2.33	2.32	2.31	2.30
	0.05	3.14	3.10	3.07	3.05	3.03	3.01	2.99	2.97	2.96	2.95
	0.025	3.96	3.91	3.87	3.83	3.80	3.77	3.74	3.72	3.70	3.68
	0.01	5.26	5.18	5.11	5.05	5.01	4.96	4.92	4.89	4.86	4.83
	0.001	9.89	9.72	9.57	9.44	9.33	9.24	9.15	9.08	9.01	8.95
10	0.1	2.32	2.30	2.28	2.27	2.26	2.24	2.23	2.22	2.22	2.21
	0.05	2.98	2.94	2.91	2.89	2.86	2.85	2.83	2.81	2.80	2.79
	0.025	3.72	3.66	3.62	3.58	3.55	3.52	3.50	3.47	3.45	3.44
	0.01	4.85	4.77	4.71	4.65	4.60	4.56	4.52	4.49	4.46	4.43
	0.001	8.75	8.59	8.45	8.32	8.22	8.13	8.05	7.98	7.91	7.86
11	0.1	2.25	2.23	2.21	2.19	2.18	2.17	2.16	2.15	2.14	2.13
	0.05	2.85	2.82	2.79	2.76	2.74	2.72	2.70	2.69	2.67	2.66
	0.025	3.53	3.47	3.43	3.39	3.36	3.33	3.30	3.28	3.26	3.24
	0.01	4.54	4.46	4.40	4.34	4.29	4.25	4.21	4.18	4.15	4.12
	0.001	7.92	7.76	7.63	7.51	7.41	7.32	7.24	7.17	7.11	7.06
12	0.1	2.19	2.17	2.15	2.13	2.12	2.10	2.09	2.08	2.08	2.07
	0.05	2.75	2.72	2.69	2.66	2.64	2.62	2.60	2.58	2.57	2.56
	0.025	3.37	3.32	3.28	3.24	3.21	3.18	3.15	3.13	3.11	3.09
	0.01	4.30	4.22	4.16	4.10	4.05	4.01	3.97	3.94	3.91	3.88
	0.001	7.29	7.14	7.00	6.89	6.79	6.71	6.63	6.57	6.51	6.45
13	0.1	2.14	2.12	2.10	2.08	2.07	2.05	2.04	2.03	2.02	2.01
	0.05	2.67	2.63	2.60	2.58	2.55	2.53	2.51	2.50	2.48	2.47
	0.025	3.25	3.20	3.15	3.12	3.08	3.05	3.03	3.00	2.98	2.96
	0.01	4.10	4.02	3.96	3.91	3.86	3.82	3.78	3.75	3.72	3.69
	0.001	6.80	6.65	6.52	6.41	6.31	6.23	6.16	6.09	6.03	5.98

F distribution critical values

Denominator degrees of freedom	p	10	11	12	13	14	15	16	17	18	19
		Degrees of freedom in the numerator									
14	0.1	2.10	2.07	2.05	2.04	2.02	2.01	2.00	1.99	1.98	1.97
	0.05	2.60	2.57	2.53	2.51	2.48	2.46	2.44	2.43	2.41	2.40
	0.025	3.15	3.09	3.05	3.01	2.98	2.95	2.92	2.90	2.88	2.86
	0.01	3.94	3.86	3.80	3.75	3.70	3.66	3.62	3.59	3.56	3.53
	0.001	6.40	6.26	6.13	6.02	5.93	5.85	5.78	5.71	5.66	5.60
15	0.1	2.06	2.04	2.02	2.00	1.99	1.97	1.96	1.95	1.94	1.93
	0.05	2.54	2.51	2.48	2.45	2.42	2.40	2.38	2.37	2.35	2.34
	0.025	3.06	3.01	2.96	2.92	2.89	2.86	2.84	2.81	2.79	2.77
	0.01	3.80	3.73	3.67	3.61	3.56	3.52	3.49	3.45	3.42	3.40
	0.001	6.08	5.94	5.81	5.71	5.62	5.54	5.46	5.40	5.35	5.29
16	0.1	2.03	2.01	1.99	1.97	1.95	1.94	1.93	1.92	1.91	1.90
	0.05	2.49	2.46	2.42	2.40	2.37	2.35	2.33	2.32	2.30	2.29
	0.025	2.99	2.93	2.89	2.85	2.82	2.79	2.76	2.74	2.72	2.70
	0.01	3.69	3.62	3.55	3.50	3.45	3.41	3.37	3.34	3.31	3.28
	0.001	5.81	5.67	5.55	5.44	5.35	5.27	5.20	5.14	5.09	5.04
17	0.1	2.00	1.98	1.96	1.94	1.93	1.91	1.90	1.89	1.88	1.87
	0.05	2.45	2.41	2.38	2.35	2.33	2.31	2.29	2.27	2.26	2.24
	0.025	2.92	2.87	2.82	2.79	2.75	2.72	2.70	2.67	2.65	2.63
	0.01	3.59	3.52	3.46	3.40	3.35	3.31	3.27	3.24	3.21	3.19
	0.001	5.58	5.44	5.32	5.22	5.13	5.05	4.99	4.92	4.87	4.82
18	0.1	1.98	1.95	1.93	1.92	1.90	1.89	1.87	1.86	1.85	1.84
	0.05	2.41	2.37	2.34	2.31	2.29	2.27	2.25	2.23	2.22	2.20
	0.025	2.87	2.81	2.77	2.73	2.70	2.67	2.64	2.62	2.60	2.58
	0.01	3.51	3.43	3.37	3.32	3.27	3.23	3.19	3.16	3.13	3.10
	0.001	5.39	5.25	5.13	5.03	4.94	4.87	4.80	4.74	4.68	4.63
19	0.1	1.96	1.93	1.91	1.89	1.88	1.86	1.85	1.84	1.83	1.82
	0.05	2.38	2.34	2.31	2.28	2.26	2.23	2.21	2.20	2.18	2.17
	0.025	2.82	2.76	2.72	2.68	2.65	2.62	2.59	2.57	2.55	2.53
	0.01	3.43	3.36	3.30	3.24	3.19	3.15	3.12	3.08	3.05	3.03
	0.001	5.22	5.08	4.97	4.87	4.78	4.70	4.64	4.58	4.52	4.47
20	0.1	1.94	1.91	1.89	1.87	1.86	1.84	1.83	1.82	1.81	1.80
	0.05	2.35	2.31	2.28	2.25	2.22	2.20	2.18	2.17	2.15	2.14
	0.025	2.77	2.72	2.68	2.64	2.60	2.57	2.55	2.52	2.50	2.48
	0.01	3.37	3.29	3.23	3.18	3.13	3.09	3.05	3.02	2.99	2.96
	0.001	5.08	4.94	4.82	4.72	4.64	4.56	4.49	4.44	4.38	4.33
21	0.1	1.92	1.90	1.87	1.86	1.84	1.83	1.81	1.80	1.79	1.78
	0.05	2.32	2.28	2.25	2.22	2.20	2.18	2.16	2.14	2.12	2.11
	0.025	2.73	2.68	2.64	2.60	2.56	2.53	2.51	2.48	2.46	2.44
	0.01	3.31	3.24	3.17	3.12	3.07	3.03	2.99	2.96	2.93	2.90
	0.001	4.95	4.81	4.70	4.60	4.51	4.44	4.37	4.31	4.26	4.21
22	0.1	1.90	1.88	1.86	1.84	1.83	1.81	1.80	1.79	1.78	1.77
	0.05	2.30	2.26	2.23	2.20	2.17	2.15	2.13	2.11	2.10	2.08
	0.025	2.70	2.65	2.60	2.56	2.53	2.50	2.47	2.45	2.43	2.41
	0.01	3.26	3.18	3.12	3.07	3.02	2.98	2.94	2.91	2.88	2.85
	0.001	4.83	4.70	4.58	4.49	4.40	4.33	4.26	4.20	4.15	4.10
23	0.1	1.89	1.87	1.84	1.83	1.81	1.80	1.78	1.77	1.76	1.75
	0.05	2.27	2.24	2.20	2.18	2.15	2.13	2.11	2.09	2.08	2.06
	0.025	2.67	2.62	2.57	2.53	2.50	2.47	2.44	2.42	2.39	2.37
	0.01	3.21	3.14	3.07	3.02	2.97	2.93	2.89	2.86	2.83	2.80
	0.001	4.73	4.60	4.48	4.39	4.30	4.23	4.16	4.10	4.05	4.00
24	0.1	1.88	1.85	1.83	1.81	1.80	1.78	1.77	1.76	1.75	1.74
	0.05	2.25	2.22	2.18	2.15	2.13	2.11	2.09	2.07	2.05	2.04
	0.025	2.64	2.59	2.54	2.50	2.47	2.44	2.41	2.39	2.36	2.35
	0.01	3.17	3.09	3.03	2.98	2.93	2.89	2.85	2.82	2.79	2.76
	0.001	4.64	4.51	4.39	4.30	4.21	4.14	4.07	4.02	3.96	3.92
25	0.1	1.87	1.84	1.82	1.80	1.79	1.77	1.76	1.75	1.74	1.73
	0.05	2.24	2.20	2.16	2.14	2.11	2.09	2.07	2.05	2.04	2.02
	0.025	2.61	2.56	2.51	2.48	2.44	2.41	2.38	2.36	2.34	2.32
	0.01	3.13	3.06	2.99	2.94	2.89	2.85	2.81	2.78	2.75	2.72
	0.001	4.56	4.42	4.31	4.22	4.13	4.06	3.99	3.94	3.88	3.84

F distribution critical values

Denominator degrees of freedom		Degrees of freedom in the numerator									
	p	10	11	12	13	14	15	16	17	18	19
	0.1	1.86	1.83	1.81	1.79	1.77	1.76	1.75	1.73	1.72	1.71
	0.05	2.22	2.18	2.15	2.12	2.09	2.07	2.05	2.03	2.02	2.00
26	0.025	2.59	2.54	2.49	2.45	2.42	2.39	2.36	2.34	2.31	2.29
	0.01	3.09	3.02	2.96	2.90	2.86	2.81	2.78	2.75	2.72	2.69
	0.001	4.48	4.35	4.24	4.14	4.06	3.99	3.92	3.86	3.81	3.77
	0.1	1.85	1.82	1.80	1.78	1.76	1.75	1.74	1.72	1.71	1.70
	0.05	2.20	2.17	2.13	2.10	2.08	2.06	2.04	2.02	2.00	1.99
27	0.025	2.57	2.51	2.47	2.43	2.39	2.36	2.34	2.31	2.29	2.27
	0.01	3.06	2.99	2.93	2.87	2.82	2.78	2.75	2.71	2.68	2.66
	0.001	4.41	4.28	4.17	4.08	3.99	3.92	3.86	3.80	3.75	3.70
	0.1	1.84	1.81	1.79	1.77	1.75	1.74	1.73	1.71	1.70	1.69
	0.05	2.19	2.15	2.12	2.09	2.06	2.04	2.02	2.00	1.99	1.97
28	0.025	2.55	2.49	2.45	2.41	2.37	2.34	2.32	2.29	2.27	2.25
	0.01	3.03	2.96	2.90	2.84	2.79	2.75	2.72	2.68	2.65	2.63
	0.001	4.35	4.22	4.11	4.01	3.93	3.86	3.80	3.74	3.69	3.64
	0.1	1.83	1.80	1.78	1.76	1.75	1.73	1.72	1.71	1.69	1.68
	0.05	2.18	2.14	2.10	2.08	2.05	2.03	2.01	1.99	1.97	1.96
29	0.025	2.53	2.48	2.43	2.39	2.36	2.32	2.30	2.27	2.25	2.23
	0.01	3.00	2.93	2.87	2.81	2.77	2.73	2.69	2.66	2.63	2.60
	0.001	4.29	4.16	4.05	3.96	3.88	3.80	3.74	3.68	3.63	3.59
	0.1	1.82	1.79	1.77	1.75	1.74	1.72	1.71	1.70	1.69	1.68
	0.05	2.16	2.13	2.09	2.06	2.04	2.01	1.99	1.98	1.96	1.95
30	0.025	2.51	2.46	2.41	2.37	2.34	2.31	2.28	2.26	2.23	2.21
	0.01	2.98	2.91	2.84	2.79	2.74	2.70	2.66	2.63	2.60	2.57
	0.001	4.24	4.11	4.00	3.91	3.82	3.75	3.69	3.63	3.58	3.53
	0.1	1.76	1.74	1.71	1.70	1.68	1.66	1.65	1.64	1.62	1.61
	0.05	2.08	2.04	2.00	1.97	1.95	1.92	1.90	1.89	1.87	1.85
40	0.025	2.39	2.33	2.29	2.25	2.21	2.18	2.15	2.13	2.11	2.09
	0.01	2.80	2.73	2.66	2.61	2.56	2.52	2.48	2.45	2.42	2.39
	0.001	3.87	3.75	3.64	3.55	3.47	3.40	3.34	3.28	3.23	3.19
	0.1	1.73	1.70	1.68	1.66	1.64	1.63	1.61	1.60	1.59	1.58
	0.05	2.03	1.99	1.95	1.92	1.89	1.87	1.85	1.83	1.81	1.80
50	0.025	2.32	2.26	2.22	2.18	2.14	2.11	2.08	2.06	2.03	2.01
	0.01	2.70	2.63	2.56	2.51	2.46	2.42	2.38	2.35	2.32	2.29
	0.001	3.67	3.55	3.44	3.35	3.27	3.20	3.14	3.09	3.04	2.99
	0.1	1.71	1.68	1.66	1.64	1.62	1.60	1.59	1.58	1.56	1.55
	0.05	1.99	1.95	1.92	1.89	1.86	1.84	1.82	1.80	1.78	1.76
60	0.025	2.27	2.22	2.17	2.13	2.09	2.06	2.03	2.01	1.98	1.96
	0.01	2.63	2.56	2.50	2.44	2.39	2.35	2.31	2.28	2.25	2.22
	0.001	3.54	3.42	3.32	3.23	3.15	3.08	3.02	2.96	2.91	2.87
	0.1	1.66	1.64	1.61	1.59	1.57	1.56	1.54	1.53	1.52	1.50
	0.05	1.93	1.89	1.85	1.82	1.79	1.77	1.75	1.73	1.71	1.69
100	0.025	2.18	2.12	2.08	2.04	2.00	1.97	1.94	1.91	1.89	1.87
	0.01	2.50	2.43	2.37	2.31	2.27	2.22	2.19	2.15	2.12	2.09
	0.001	3.30	3.18	3.07	2.99	2.91	2.84	2.78	2.73	2.68	2.63
	0.1	1.63	1.60	1.58	1.56	1.54	1.52	1.51	1.49	1.48	1.47
	0.05	1.88	1.84	1.80	1.77	1.74	1.72	1.69	1.67	1.66	1.64
200	0.025	2.11	2.06	2.01	1.97	1.93	1.90	1.87	1.84	1.82	1.80
	0.01	2.41	2.34	2.27	2.22	2.17	2.13	2.09	2.06	2.03	2.00
	0.001	3.12	3.00	2.90	2.82	2.74	2.67	2.61	2.56	2.51	2.46
	0.1	1.61	1.58	1.55	1.53	1.51	1.49	1.48	1.46	1.45	1.44
	0.05	1.84	1.80	1.76	1.73	1.70	1.68	1.65	1.63	1.61	1.60
1000	0.025	2.06	2.01	1.96	1.92	1.88	1.85	1.82	1.79	1.77	1.74
	0.01	2.34	2.27	2.20	2.15	2.10	2.06	2.02	1.98	1.95	1.92
	0.001	2.99	2.87	2.77	2.69	2.61	2.54	2.48	2.43	2.38	2.34

Probabilities for the standard normal distribution

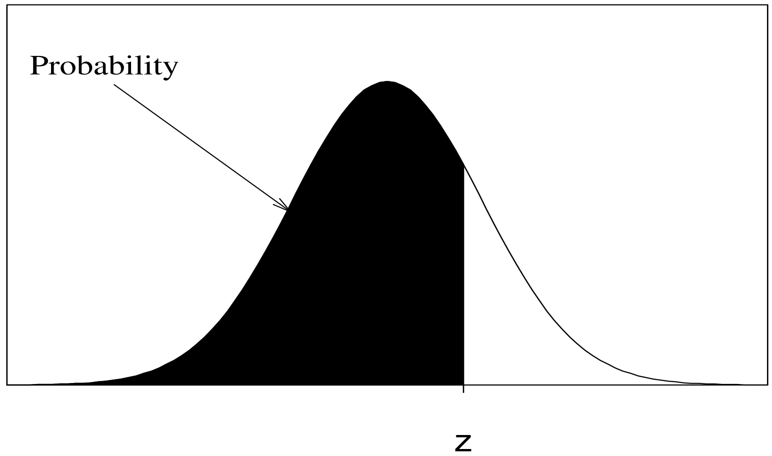
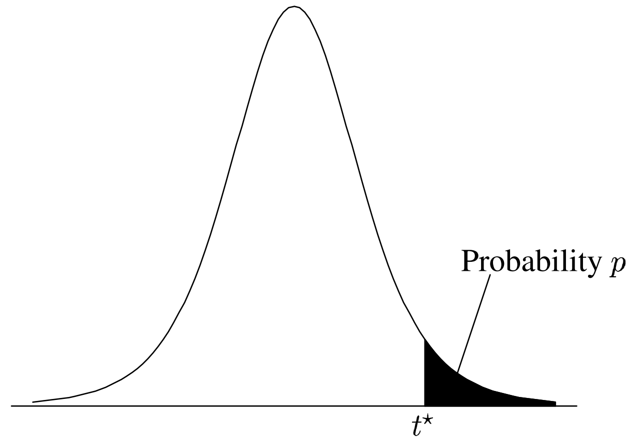


Table entry for z is the probability lying to the left of z

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Probabilities for the t-distribution

Table entry for p and C is the point $t^\star$ with probability p lying above it and probability C lying between $-t^\star$ and $t^\star$



	Tail probability p											
df	.25	.2	.15	.1	.05	.025	.02	.01	.005	.0025	.001	.0005
1	1.000	1.376	1.963	3.078	6.314	12.706	15.895	31.821	63.657	127.321	318.309	636.619
2	0.816	1.061	1.386	1.886	2.920	4.303	4.849	6.965	9.925	14.089	22.327	31.599
3	0.765	0.978	1.250	1.638	2.353	3.182	3.482	4.541	5.841	7.453	10.215	12.924
4	0.741	0.941	1.190	1.533	2.132	2.776	2.999	3.747	4.604	5.598	7.173	8.610
5	0.727	0.920	1.156	1.476	2.015	2.571	2.757	3.365	4.032	4.773	5.893	6.869
6	0.718	0.906	1.134	1.440	1.943	2.447	2.612	3.143	3.707	4.317	5.208	5.959
7	0.711	0.896	1.119	1.415	1.895	2.365	2.517	2.998	3.499	4.029	4.785	5.408
8	0.706	0.889	1.108	1.397	1.860	2.306	2.449	2.896	3.355	3.833	4.501	5.041
9	0.703	0.883	1.100	1.383	1.833	2.262	2.398	2.821	3.250	3.690	4.297	4.781
10	0.700	0.879	1.093	1.372	1.812	2.228	2.359	2.764	3.169	3.581	4.144	4.587
11	0.697	0.876	1.088	1.363	1.796	2.201	2.328	2.718	3.106	3.497	4.025	4.437
12	0.695	0.873	1.083	1.356	1.782	2.179	2.303	2.681	3.055	3.428	3.930	4.318
13	0.694	0.870	1.079	1.350	1.771	2.160	2.282	2.650	3.012	3.372	3.852	4.221
14	0.692	0.868	1.076	1.345	1.761	2.145	2.264	2.624	2.977	3.326	3.787	4.140
15	0.691	0.866	1.074	1.341	1.753	2.131	2.249	2.602	2.947	3.286	3.733	4.073
16	0.690	0.865	1.071	1.337	1.746	2.120	2.235	2.583	2.921	3.252	3.686	4.015
17	0.689	0.863	1.069	1.333	1.740	2.110	2.224	2.567	2.898	3.222	3.646	3.965
18	0.688	0.862	1.067	1.330	1.734	2.101	2.214	2.552	2.878	3.197	3.610	3.922
19	0.688	0.861	1.066	1.328	1.729	2.093	2.205	2.539	2.861	3.174	3.579	3.883
20	0.687	0.860	1.064	1.325	1.725	2.086	2.197	2.528	2.845	3.153	3.552	3.850
21	0.686	0.859	1.063	1.323	1.721	2.080	2.189	2.518	2.831	3.135	3.527	3.819
22	0.686	0.858	1.061	1.321	1.717	2.074	2.183	2.508	2.819	3.119	3.505	3.792
23	0.685	0.858	1.060	1.319	1.714	2.069	2.177	2.500	2.807	3.104	3.485	3.768
24	0.685	0.857	1.059	1.318	1.711	2.064	2.172	2.492	2.797	3.091	3.467	3.745
25	0.684	0.856	1.058	1.316	1.708	2.060	2.167	2.485	2.787	3.078	3.450	3.725
26	0.684	0.856	1.058	1.315	1.706	2.056	2.162	2.479	2.779	3.067	3.435	3.707
27	0.684	0.855	1.057	1.314	1.703	2.052	2.158	2.473	2.771	3.057	3.421	3.690
28	0.683	0.855	1.056	1.313	1.701	2.048	2.154	2.467	2.763	3.047	3.408	3.674
29	0.683	0.854	1.055	1.311	1.699	2.045	2.150	2.462	2.756	3.038	3.396	3.659
30	0.683	0.854	1.055	1.310	1.697	2.042	2.147	2.457	2.750	3.030	3.385	3.646
40	0.681	0.851	1.050	1.303	1.684	2.021	2.123	2.423	2.704	2.971	3.307	3.551
50	0.679	0.849	1.047	1.299	1.676	2.009	2.109	2.403	2.678	2.937	3.261	3.496
60	0.679	0.848	1.045	1.296	1.671	2.000	2.099	2.390	2.660	2.915	3.232	3.460
80	0.678	0.846	1.043	1.292	1.664	1.990	2.088	2.374	2.639	2.887	3.195	3.416
100	0.677	0.845	1.042	1.290	1.660	1.984	2.081	2.364	2.626	2.871	3.174	3.390
1000	0.675	0.842	1.037	1.282	1.646	1.962	2.056	2.330	2.581	2.813	3.098	3.300
∞	0.674	0.842	1.036	1.282	1.645	1.960	2.054	2.326	2.576	2.807	3.090	3.291
	50%	60%	70 %	80%	90%	95%	96%	98%	99%	99.5%	99.8%	99.9%
	Confidence level C											