EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2020	MATH3071-WE01

Title:
Decision Theory III

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. Show your working and explain your reasoning.

Revision:	

ED01/2020

Q1 **1.1** The managers of a hospital wish to determine how many ambulances it should have and how to staff them. Each ambulance may be staffed by paramedics or emergency medical technicians. Paramedics are considered to provide a better service and are paid higher salaries. Budget limitations force a choice between the following two alternatives:

- Alternative 1: Four ambulances, two staffed by emergency medical technicians and two by paramedics.
- Alternative 2: Three ambulances, all staffed by paramedics.

The hospital managers believe that the following two attributes determine the overall satisfaction with the ambulance service:

- Attribute T: Time until an ambulance reaches a patient.
- Attribute C: Percentage of ambulance calls handled by paramedics.

Assume that the managers' multi-attribute utility function $u(T, C)$ exhibits mutual utility independence. Assume further that the time for an ambulance to reach a patient can range from 0 to 30 minutes, and that for any value $t \in [0, 30]$ of T (in minutes) and any value $c \in [0, 100]$ of C, the marginal utility functions are of the following forms up to a positive linear transformation:

$$u(t, 0) = -t^2 \quad \text{and} \quad u(30, c) = c^2$$

Clearly, the worst possible scenario is $(t, c) = (30, 0)$, so set $u(30, 0) = 0$. Suppose that the managers consider that $(30, 100) \sim^* (0, 0)$ and that $(10, 25) \sim^* (20, 60)$.

Assume:

- if an ambulance is immediately available when a call comes in, then the ambulance will arrive in 5 minutes;
- if an ambulance is not immediately available when a call comes in, then it will arrive in 20 minutes;
- with three ambulances, one will be immediately available for 60% of all calls;
- with four ambulances, one will be immediately available for 80% of all calls;
- for Alternative 1, 50% of calls will be handled by paramedics.

(i) Determine the hospital managers' multi-attribute utility function $u(T, C)$.

(ii) Explain which alternative they should choose.

(iii) By considering two gambles involving the attribute value pairs $(0, 0)$, $(30, 0)$, $(0, 100)$, and $(30, 100)$, discuss whether these two attributes can be considered to be complementary or substitutable.

1.2 A company wants to hire a new member of staff. For the first stage of the decision process, a panel of four people (A - D) will judge the candidates and combine their preferences into a group preference order. There are five candidates (C1 - C5). In order to combine their individual preferences into a group preference order, they decide to use the following general procedure: Each panel member gives a non-negative integer score to each candidate, with the total sum of the scores per panel member at most 10. The sums of the scores per candidate determine the group preference ranking in the natural way, with a higher total representing higher group preference and equal score representing equal group preference. The scores per panel member are as follows

	C1	C2	C3	C4	C5
A	2	2	3	1	2
B	10	0	0	0	0
C	0	5	5	0	0
D	2	0	6	1	1

(i) Derive the group preference order for the decision problem above. Discuss the general procedure used from the perspective of Arrow's theorem, in particular explain for each of the four axioms if they are satisfied. Illustrate any unsatisfied axioms using this decision problem.

(ii) Explain whether or not, in this procedure, it can make sense for a panel member to award scores summing up to a total that is less than 10.

Q2 You wish to buy a car and have a choice between a new car costing £8,000 and a second-hand car costing £5,500. If the car that you buy turns out to be unreliable then you will have to hire a car whenever it is being repaired. The guarantee on the new car covers both repair costs and hire charges for a period of two years. There is no guarantee on the second-hand car. You know that second-hand cars generally are either good or bad. You believe that the probability that the second-hand car you consider buying is good is 0.4, and that in this case it will cost you £1,500 on repairs and car hire charges over the next two years. With probability 0.6 this car will turn out to be bad, in which case it will cost you £3,500 for repairs and car hire charges over the next two years.

A roadside repair service can test the second-hand car at no financial cost, but this could lead to this car being sold while the repair service are arranging a road test; this would happen with probability 1/3 and in this case you would have to buy the new car. The road test will give a 'satisfactory' or 'unsatisfactory' report with the following probabilities: if the car is bad, the report will be 'satisfactory' with probability 0.1 and 'unsatisfactory' with probability 0.9; if the car is good the report will be 'satisfactory' with probability 0.9 and 'unsatisfactory' with probability 0.1.

Your own garage can do a similar test on the second-hand car without the risk to lose the option to buy this car, and also at no financial cost. But you do not fully trust them; your probability of a 'satisfactory' report if the car is bad is 0.5, the same as your probability for an 'unsatisfactory' report in this case; if the car is good the garage's report will certainly be 'satisfactory'.

You can buy the new car or the second-hand car without any test, or go for either the repair service test or the garage's test, but not both, on the second-hand car. Assume that you want to maximise expected monetary value (so minimise total expected costs) and that no other attributes are important to you.

2.1 Represent this decision problem by a decision tree; take particular care on deriving the (conditional) probabilities and explain briefly how you calculate these. Describe the full optimal solution. Give the risk profile corresponding to the optimal solution, taking all uncertainties into account.

2.2 Suppose that you are not certain about the probability 1/3 for the event that the second-hand car will be sold whilst the repair service are arranging a road test; let this probability be $p \in [0, 1]$. Analyze the sensitivity of the optimal decision to this problem with regard to the value of p.

2.3 Suppose that a further option is possible, namely to have a perfect test on the second-hand car (so it reveals with certainty if it is a good or bad buy) without the risk for it to be sold while the test takes place. What is the maximum amount of money you would be willing to pay for this test?

2.4 In question **2.1**, there were four options for the initial decision: buy the new or second-hand car without a test, or go for one of the two tests on the second-hand car. Identify the second best option of these four, i.e. the best one if the optimal decision from question **2.1** is excluded. Give the risk profile for this second best initial decision, and compare it to the risk profile for the optimal decision in question **2.1**.

Q3 Items from a production process are either good or bad. Let w be the probability that an item is good, where the qualities of different items are independent conditional upon w. It is known that $w \in \{0.25, 0.5, 0.75\}$. Suppose that your prior probabilities for these three possible values of w are all equal to $1/3$. You wish to estimate w by $d \in \{0.25, 0.5, 0.75\}$, aiming at minimum expected value of the loss function

$$L(w,d) = \begin{cases} 0 & \text{if } w = d, \\ 1 & \text{if } w \neq d. \end{cases}$$

You will have the opportunity to sample 2 independent items. The number X of good items in this sample is Binomially distributed, $X \sim \text{Bi}(2, w)$, so the probability distribution of X is given by

$$P(X = k|w) = \binom{2}{k} w^k (1-w)^{2-k} \quad \text{for } k = 0, 1, 2.$$

3.1 Find the Bayes decision and Bayes risk of an immediate decision in this estimation problem, so without sampling. Briefly comment on your answer.

3.2 For each value $k = 0, 1, 2$, find the Bayes decision and Bayes risk after observing k good items in the sample of 2 items.

3.3 Find the Bayes risk of sampling for a sample of size 2.

3.4 Suppose that you are offered the opportunity to buy specific information as follows: for a price $\pi > 0$ (in the same units as the losses in the loss function), you are correctly given one specific value for w, of the three values which you considered possible, which is *not* its value. As you have no further information, you judge it equally likely that any of those three values will be named as being impossible, if you decide to pay for this information; in this case, your prior probabilities for each of the two remaining possible values for w become $1/2$ and you can again sample 2 items. For which values of π would you wish to buy this information (assuming that you wish to minimise overall expected loss)?

Q4 Ursula and Violet want to choose a joint activity from four options (A - D). Their utilities for these options are given in the table below. They consider the further option, of not doing a joint activity, as the status quo, both have utility 2 for this.

	A	B	C	D
Ursula	8	2	7	8
Violet	4	11	7	6

4.1 Sketch the feasible region for this bargaining problem, identify the Pareto Boundary and the status quo point.

4.2 Compute the Nash point for this problem by solving the optimisation problem in the general definition of the Nash point. Also present a graphical derivation of this Nash point, with detailed explanation of the Nash axioms used for this procedure.

4.3 Compute the equitable distribution point for this problem and identify the corresponding gamble.

4.4 Violet prefers the equitable distribution point over the Nash point as solution to this problem, Ursula prefers the Nash point. To resolve this, they agree to decide on which of these two solutions to adopt by tossing a fair coin. Discuss this method to solve such a bargaining problem in detail, so as a general procedure. For each of the six Nash axioms and the monotonicity axiom corresponding to the equitable distribution point, explain whether or not the axiom is satisfied by the solution from this procedure.

Q5 **5.1** (i) Four friends (A - D) try to decide on a joint activity for an afternoon, they consider four options: visiting a Garden (G), going for a Swim (S), afternoon Tea (T), a nice Walk (W). The option not to do something (N) is of course also available. Their utilities for these options are given in the table below.

	G	S	T	W	N
A	8	-2	6	2	-2
B	0.3	1	0	1	0
C	4	6	1	4	1
D	2	1	3	5	1

Apply Harsanyi's theorem of utilitarianism to create a group preference ranking over these options. Briefly explain why the inclusion of option N with the utilities as given is important for the application of this theorem.

(ii) Five people who share an office have to decide on the room temperature. They decide that each writes down their ideal room temperature, then they set the room temperature at the median of these five values. Assume that each individual's preference ordering for room temperature is single-peaked. Show that this method to decide on the room temperature is attractive from the perspective of the simple majority rule to compare different temperatures. Discuss whether or not an individual can benefit from not fairly reporting their ideal room temperature, both if they have no idea about the ideal temperatures of the others and if they happen to know these values.

5.2 Consider the following two person $m \times n$ zero-sum game: R chooses strategy (row) R_i with probability p_i, for $i = 1, \dots, m$; C chooses strategy (column) C_j with probability q_j, for $j = 1, \dots, n$. For these specific chosen strategies, the payoff to R is a_{ij} and the payoff to C is $-a_{ij}$. Both players aim to maximise their expected payoff.

(i) Provide a detailed general description and justification of a graphical method to find the minimax solution for such a game for the case $m = 2$ and $n \geq 2$. Apply this method to solve the zero-sum game with the following payoff table, so derive the minimax strategies and the value of the game.

	C_1	C_2	C_3	C_4
R_1	3	1	4	0
R_2	1	2	0	5

(ii) A similar graphical method can be used for the case $m \geq 2$ and $n = 2$. Describe and justify this method in detail, and apply it to solve the zero-sum game with the following payoff table.

	C_1	C_2
R_1	10	2
R_2	2	10
R_3	4	8
R_4	1	12