

EXAMINATION PAPER

Examination Session: May/June

2020

Year:

Exam Code:

MATH3091-WE01

Title:

Dynamical Systems III

| Time (for guidance only): | 3 hours | |
|-------------------------------|---------|---|
| Additional Material provided: | | |
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| Materials Permitted: | | |
| | | |
| Calculators Permitted: | Yes | Models Permitted: There is no restriction on the model of calculator which may be used. |

| Instructions to Candidates: | Credit will be given for your answers to all questions. All questions carry the same marks. | | |
|-----------------------------|--|--|--|
| | Please start each question on a new page. Please write your CIS username at the top of each page. | | |
| | Show your working and explain your reasoning. | | |
| | | | |

Revision:

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Q1 Consider the following dynamical system

$$\begin{cases} x_1' = x_1 - e^{x_2} \\ x_2' = \log |x_2| \end{cases}$$

- 1.1 Find the critical points of this system.
- 1.2 Are the critical points you found in part 1.1 hyperbolic?

Now consider a one-dimensional dynamical system $x' = f(x, \mu)$ with $f(x, \mu)$ infinitely differentiable in both of its arguments.

1.3 State necessary conditions for the system to undergo a local bifurcation at a point (x_*, μ_*) .

Consider now $f(x, \mu) = (x^2 - \mu^2)(x^2 - (\mu + 2)^2).$

- 1.4 Work out the conditions under part 1.3 and find all bifurcation points.
- 1.5 Draw a bifurcation diagram. State the type of all the bifurcation points.
- Q2 2.1 Consider the ordinary differential equation

$$x''' = x'' + 2x'$$

and verify that it is equivalent to the following linear dynamical system

$$\mathbf{x}' = A \cdot \mathbf{x}$$
 $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix}$

- **2.2** Determine the matrix U such that $J(A) = U^{-1}AU$ is in Jordan normal form. Verify your answer by computing its inverse and checking that this is indeed the case.
- **2.3** Determine the phase flow of the linear system $\mathbf{x}(t)$ for arbitrary values of the initial conditions

$$\mathbf{x}_0 = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \qquad \alpha, \beta, \gamma \in \mathbb{R}$$

2.4 Using your result from part **2.3** above determine all possible solutions to the equation x''' = x'' + 2x' such that

(1)
$$x'(0) = 0$$
 and $x''(0) = 0$

(2) x(0) = 0 and x'(0) = 0



Q3 In this question we will determine the phase flow of the following dynamical system on \mathbb{R}^2 :

$$x' = \cos y \qquad \qquad y' = \sin x$$

3.1 Determine the critical points of this system. It is convenient to use a pair of integers $n, m \in \mathbb{Z}$ to label the critical points as follows

$$\mathbf{x}_{n,m} = \begin{pmatrix} x_n \\ y_m \end{pmatrix}$$

3.2 Compute the Jacobian matrix of the vector field

$$\mathbf{f} = \begin{pmatrix} \cos y \\ \sin x \end{pmatrix}$$

and use it to determine the linearized system in a neighborhood of each critical point $\mathbf{x}_{n,m}$.

- **3.3** Organize the Cartesian plane in a grid with vertices at the points $\mathbf{x}_{n,m}$. Associate to each critical point of the grid its corresponding local flow and complete the qualitative picture of the phase flow, remembering that flow lines never cross.
- **3.4** Determine the invariant sets for this system.
- ${\bf Q4}\,$ Consider a smooth planar system

$$x' = f(x, y)$$
$$y' = g(x, y)$$

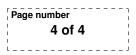
- 4.1 Define what it means for such a system to be Hamiltonian.
- 4.2 Assume the above system is Hamiltonian. Introduce new coordinates by an invertible linear map of the form

$$z = ax + by$$
$$w = cx + dy$$

Show that the system written in terms of z, w is again Hamiltonian, and find its Hamiltonian $\tilde{H}(z, w)$.

Consider now $x'' - \mu(1 - x^2)x' + x = 0.$

- 4.3 Show that this system is not Hamiltonian.
- **4.4** Let A(t) be the area of a region in the (x, x') plane which is initially defined by the interior of the square with four corners $(\pm 1, \pm 1)$ and then evolves according to the above dynamical system. Compute the initial rate of change of this area, $dA/dt|_{t=0}$.
- 4.5 Consider now a general smooth planar system with the property that all volumes shrink monotonically. Does this imply that the system has fixed points? Prove your answer.



- Q5 5.1 State the definition of the Poincaré index for planar systems.
 - **5.2** What is the behavior of the Poincaré index under smooth deformations of the dynamical system?
 - **5.3** Prove that the Poincaré index vanishes for paths which do not enclose any fixed point in their interior.

Consider the dynamical system

$$\begin{aligned} x' &= y^2 \\ y' &= xy + a \end{aligned}$$

- 5.4 Show that the Poincaré index on the unit circle around the origin vanishes.
- **5.5** For a = 0, there is a saddle point at the origin. Why does this not contradict the invariance of the index under smooth deformations?

Consider a three-dimensional dynamical system for which $H = x^2 + y^2 - z^2$ is a first integral, and for which the flow on the disc $D = \{(x, y) | x^2 + y^2 = 1\}$ points strictly outward.

5.6 Use the Poincaré index to argue for the existence of a fixed point on each surface H = c, c < 0.