Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2020	MATH3091-WE01

Title:
Dynamical Systems III

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. Show your working and explain your reasoning.

Revision:	

Q1 Consider the following dynamical system

$$\begin{cases} {x_1}' = x_1 - e^{x_2} \\ {x_2}' = \log |x_2| \end{cases}$$

1.1 Find the critical points of this system.

1.2 Are the critical points you found in part **1.1** hyperbolic?

Now consider a one-dimensional dynamical system $x' = f(x, \mu)$ with $f(x, \mu)$ infinitely differentiable in both of its arguments.

1.3 State necessary conditions for the system to undergo a local bifurcation at a point (x_*, μ_*).

Consider now $f(x, \mu) = (x^2 - \mu^2)(x^2 - (\mu + 2)^2)$.

1.4 Work out the conditions under part **1.3** and find all bifurcation points.

1.5 Draw a bifurcation diagram. State the type of all the bifurcation points.

Q2 **2.1** Consider the ordinary differential equation

$$x''' = x'' + 2x'$$

and verify that it is equivalent to the following linear dynamical system

$$\mathbf{x}' = A \cdot \mathbf{x} \qquad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix}$$

2.2 Determine the matrix U such that $J(A) = U^{-1}AU$ is in Jordan normal form. Verify your answer by computing its inverse and checking that this is indeed the case.

2.3 Determine the phase flow of the linear system $\mathbf{x}(t)$ for arbitrary values of the initial conditions

$$\mathbf{x}_0 = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \qquad \alpha, \beta, \gamma \in \mathbb{R}$$

2.4 Using your result from part **2.3** above determine all possible solutions to the equation $x''' = x'' + 2x'$ such that

(1) $x'(0) = 0$ and $x''(0) = 0$

(2) $x(0) = 0$ and $x'(0) = 0$

Q3 In this question we will determine the phase flow of the following dynamical system on $\mathbb{R}^2$:

$$x' = \cos y \qquad\qquad y' = \sin x$$

3.1 Determine the critical points of this system. It is convenient to use a pair of integers $n, m \in \mathbb{Z}$ to label the critical points as follows

$$\mathbf{x}_{n,m} = \begin{pmatrix} x_n \\ y_m \end{pmatrix}$$

3.2 Compute the Jacobian matrix of the vector field

$$\mathbf{f} = \begin{pmatrix} \cos y \\ \sin x \end{pmatrix}$$

and use it to determine the linearized system in a neighborhood of each critical point $\mathbf{x}_{n,m}$.

3.3 Organize the Cartesian plane in a grid with vertices at the points $\mathbf{x}_{n,m}$. Associate to each critical point of the grid its corresponding local flow and complete the qualitative picture of the phase flow, remembering that flow lines never cross.

3.4 Determine the invariant sets for this system.

Q4 Consider a smooth planar system

$$x' = f(x, y)$$
$$y' = g(x, y)$$

4.1 Define what it means for such a system to be Hamiltonian.

4.2 Assume the above system is Hamiltonian. Introduce new coordinates by an invertible linear map of the form

$$z = ax + by$$
$$w = cx + dy$$

Show that the system written in terms of z, w is again Hamiltonian, and find its Hamiltonian $\tilde{H}(z, w)$.

Consider now $x'' - \mu(1 - x^2)x' + x = 0$.

4.3 Show that this system is not Hamiltonian.

4.4 Let $A(t)$ be the area of a region in the (x, x') plane which is initially defined by the interior of the square with four corners $(\pm 1, \pm 1)$ and then evolves according to the above dynamical system. Compute the initial rate of change of this area, $dA/dt|_{t=0}$.

4.5 Consider now a general smooth planar system with the property that all volumes shrink monotonically. Does this imply that the system has fixed points? Prove your answer.

Q5 **5.1** State the definition of the Poincaré index for planar systems.

5.2 What is the behavior of the Poincaré index under smooth deformations of the dynamical system?

5.3 Prove that the Poincaré index vanishes for paths which do not enclose any fixed point in their interior.

Consider the dynamical system

$$\begin{aligned} x' &= y^2 \\ y' &= xy + a \end{aligned} .$$

5.4 Show that the Poincaré index on the unit circle around the origin vanishes.

5.5 For $a = 0$, there is a saddle point at the origin. Why does this not contradict the invariance of the index under smooth deformations?

Consider a three-dimensional dynamical system for which $H = x^2 + y^2 - z^2$ is a first integral, and for which the flow on the disc $D = \{(x, y)|x^2 + y^2 = 1\}$ points strictly outward.

5.6 Use the Poincaré index to argue for the existence of a fixed point on each surface $H = c$, $c < 0$.