Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2020	MATH3101-WE01

Title: Continuum Mechanics III

Time (for guidance only):	3 hours	
Additional Material provided:	Formula sheet	
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. Show your working and explain your reasoning.

Revision:	

ED01/2020

Q1 **1.1** Consider the flow $\boldsymbol{u} = -\alpha x\boldsymbol{e}_x - \alpha y\boldsymbol{e}_y$, where α is a positive constant.

(i) Find the path of a particle initially located at $\boldsymbol{x} = (a, b)$.

(ii) Consider the material curve γ_t defined at $t = 0$ by $x^2 + y^2 = A^2$ where $A \in \mathbb{R}$. Find the equation for γ_t when $t > 0$.

(iii) How does the area enclosed by γ_t change with time?

1.2 An unforced incompressible Newtonian viscous fluid of viscosity μ fills a cylinder of radius R, which is rotating at constant angular velocity Ω.

(i) Find the equations and boundary conditions that must be satisfied by a solution of the form $\boldsymbol{u} = u(r)\boldsymbol{e}_\theta$, $p = p(r)$.

(ii) Solve these to find smooth solutions for $u(r)$ and $p(r)$.

(iii) If instead the fluid were inviscid ($\mu = 0$), what would be the possible $u(r)$?

Q2 In cylindrical coordinates, let us model a whirlpool of water with the steady velocity field

$$\boldsymbol{u} = \begin{cases} \Omega r\boldsymbol{e}_\theta, & r < a, \\ \dfrac{\Omega a^2}{r}\boldsymbol{e}_\theta, & r > a, \end{cases}$$

where a and Ω are positive constants. The fluid is subject to a gravitational force $-g\boldsymbol{e}_z$.

2.1 Compute the vorticity $\boldsymbol{\omega}(r)$ in the inner and outer regions. Is it continuous at $r = a$?

2.2 Find a *continuous* stream function $\psi(r)$ such that $\boldsymbol{u} = \nabla \times (\psi\boldsymbol{e}_z)$.

2.3 Show that $\boldsymbol{u}$ is a solution of the steady state, incompressible Euler equations and solve for the pressure when $p = p_0$ at $r = z = 0$.

2.4 If, in addition, p_0 is the pressure of the air above (and on the surface of) the water, find the curve $z(r)$ that describes the shape of the water's surface.

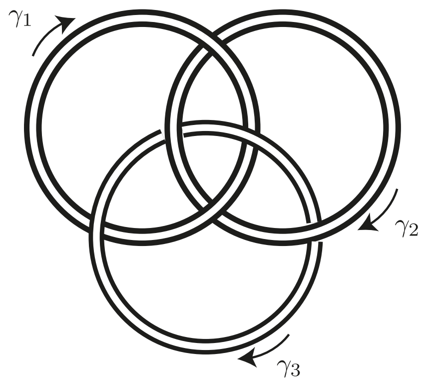
Q3 Consider an unforced, incompressible, ideal fluid.

3.1 Write down the momentum and vorticity equations in Lagrangian form.

3.2 If D_t is a simply-connected material volume in this fluid, show that the helicity $H = \displaystyle\int_V \boldsymbol{u} \cdot \boldsymbol{\omega}\, \mathrm{d}V$ obeys

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \oint_{\partial D_t} \left(\frac{1}{2}|\boldsymbol{u}|^2 - \frac{p}{\rho_0}\right) \boldsymbol{\omega} \cdot \boldsymbol{n}\, \mathrm{d}S.$$

3.3 Now suppose V contains the following configuration of three linked line vortices, with circulations C_1, C_2, C_3. The arrows indicate the direction of vorticity.



Assuming the line vortices to be infinitesimally thin, with $\boldsymbol{\omega} = \boldsymbol{0}$ everywhere else in V, determine the helicity.

Q4 An organ pipe comprises a rectangular tube of square cross-section $\{-a < x < a, -a < y < a\}$ and length b. It is closed at the bottom ($z = 0$) and open at the top ($z = b$). We consider linear sound waves inside the pipe, described by $\boldsymbol{u} = \nabla\phi$ where

$$\frac{\partial^2 \phi}{\partial t^2} = c_0^2 \Delta\phi. \qquad (*)$$

4.1 Write down the appropriate boundary conditions for ϕ on all six boundaries.

4.2 By assuming a solution of the form $\phi(x, y, z, t) = X(x)Y(y)Z(z)\mathrm{e}^{-i\omega t}$, show that the equation (*) can be separated into three ODEs

$$Z'' = -k_z^2 Z, \qquad X'' = -k_x^2 X, \qquad Y'' = -k_y^2 Y$$

and find k_y as a function of k_x and k_z.

4.3 Hence find the possible values of ω for these standing waves.

4.4 What is the fundamental frequency $\dfrac{\omega}{2\pi}$ of the pipe?

4.5 Some organs are better modelled by a tube that is open at both ends. What would you expect to be the fundamental frequency in that case?

Q5 A "self-gravitating" barotropic ideal fluid feels a gravitational body force of the form $\boldsymbol{f} = -\nabla\Phi$ with $\Phi = \Phi(x, t)$.

5.1 Write down the compressible 1D Euler equations for this fluid assuming $\boldsymbol{u} = u(x, t)\boldsymbol{e}_x$, $p = p(x, t)$ and $\rho = \rho(x, t)$.

5.2 If the fluid is initially at rest with constant density ρ_0, constant pressure p_0, and $\Phi_0 = 0$, show that small perturbations $\boldsymbol{u} = u(x, t)\boldsymbol{e}_x$, $\rho(x, t)$, $p(x, t)$, $\Phi(x, t)$ satisfy the linear equations

$$\frac{\partial \rho}{\partial t} = -\rho_0 \frac{\partial u}{\partial x},$$
$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} - \frac{\partial \Phi}{\partial x}.$$

5.3 If the perturbed gravitational potential is determined from the density perturbation by the Poisson equation

$$\frac{\partial^2 \Phi}{\partial x^2} = 4\pi G\rho,$$

with G constant, show that the system reduces to

$$\frac{\partial^2 \rho}{\partial t^2} = c_0^2 \frac{\partial^2 \rho}{\partial x^2} + 4\pi G\rho_0\rho$$

where c_0 is the sound speed.

5.4 By trying solutions of the form $\rho = \exp\left[i(kx - \omega t)\right]$, determine the condition for linear instability.

5.5 Give a brief physical interpretation of the result in **5.4**.

FORMULA SHEET for MATH 3101/4081 : CONTINUUM MECHANICS

Some vector identities:

$$\nabla \cdot \big(f\boldsymbol{A}\big) = \big(\nabla f\big) \cdot \boldsymbol{A} + f\nabla \cdot \boldsymbol{A} \tag{1}$$

$$\nabla \times \big(f\boldsymbol{A}\big) = \big(\nabla f\big) \times \boldsymbol{A} + f\nabla \times \boldsymbol{A} \tag{2}$$

$$\nabla \times \big(\nabla \times \boldsymbol{A}\big) = \nabla\big(\nabla \cdot \boldsymbol{A}\big) - \Delta\boldsymbol{A} \tag{3}$$

$$\big(\boldsymbol{A} \cdot \nabla\big)\boldsymbol{A} = \tfrac{1}{2}\nabla|\boldsymbol{A}|^2 - \boldsymbol{A} \times \big(\nabla \times \boldsymbol{A}\big) \tag{4}$$

$$\nabla \cdot \big(\boldsymbol{A} \times \boldsymbol{B}\big) = \boldsymbol{B} \cdot \big(\nabla \times \boldsymbol{A}\big) - \boldsymbol{A} \cdot \big(\nabla \times \boldsymbol{B}\big) \tag{5}$$

$$\nabla \times \big(\boldsymbol{A} \times \boldsymbol{B}\big) = \big(\boldsymbol{B} \cdot \nabla\big)\boldsymbol{A} - \big(\boldsymbol{A} \cdot \nabla\big)\boldsymbol{B} + \boldsymbol{A}\nabla \cdot \boldsymbol{B} - \boldsymbol{B}\nabla \cdot \boldsymbol{A} \tag{6}$$

$$\nabla\big(\boldsymbol{A} \cdot \boldsymbol{B}\big) = \big(\boldsymbol{A} \cdot \nabla\big)\boldsymbol{B} + \big(\boldsymbol{B} \cdot \nabla\big)\boldsymbol{A} + \boldsymbol{A} \times \big(\nabla \times \boldsymbol{B}\big) + \boldsymbol{B} \times \big(\nabla \times \boldsymbol{A}\big) \tag{7}$$

In cylindrical coordinates (r, θ, z):

$$\nabla f = \operatorname{grad} f = \boldsymbol{e}_r\partial_r f + \frac{\boldsymbol{e}_\theta}{r}\partial_\theta f + \boldsymbol{e}_z\partial_z f \tag{8}$$

$$\nabla\cdot\boldsymbol{A} = \operatorname{div}\boldsymbol{A} = \frac{1}{r}\partial_r(rA_r) + \frac{1}{r}\partial_\theta A_\theta + \partial_z A_z \tag{9}$$

$$\nabla\times\boldsymbol{A} = \operatorname{rot}\boldsymbol{A} = \Big(\frac{1}{r}\partial_\theta A_z - \partial_z A_\theta\Big)\boldsymbol{e}_r + \big(\partial_z A_r - \partial_r A_z\big)\boldsymbol{e}_\theta + \frac{1}{r}\big(\partial_r(rA_\theta) - \partial_\theta A_r\big)\boldsymbol{e}_z \tag{10}$$

$$\Delta f = \frac{1}{r}\partial_r\big(r\,\partial_r f\big) + \frac{1}{r^2}\partial_{\theta\theta} f + \partial_{zz} f \tag{11}$$

$$\Delta\boldsymbol{A} = \Big(\Delta A_r - \frac{1}{r^2}A_r - \frac{2}{r^2}\partial_\theta A_\theta\Big)\boldsymbol{e}_r + \Big(\Delta A_\theta + \frac{2}{r^2}\partial_\theta A_r - \frac{1}{r^2}A_\theta\Big)\boldsymbol{e}_\theta + \boldsymbol{e}_z\Delta A_z \tag{12}$$

$$\big(\boldsymbol{B}\cdot\nabla\big)\boldsymbol{A} = \boldsymbol{e}_r\Big(\boldsymbol{B}\cdot\nabla A_r - \frac{B_\theta A_\theta}{r}\Big) + \boldsymbol{e}_\theta\Big(\boldsymbol{B}\cdot\nabla A_\theta + \frac{B_\theta A_r}{r}\Big) + \boldsymbol{e}_z\,\boldsymbol{B}\cdot\nabla A_z \tag{13}$$

In spherical coordinates (r, θ, ϕ):

$$\nabla f = \operatorname{grad} f = \boldsymbol{e}_r\partial_r f + \frac{\boldsymbol{e}_\theta}{r}\partial_\theta f + \frac{\boldsymbol{e}_\phi}{r\sin\theta}\partial_\phi f \tag{14}$$

$$\nabla\cdot\boldsymbol{A} = \operatorname{div}\boldsymbol{A} = \frac{1}{r^2}\partial_r\big(r^2 A_r\big) + \frac{1}{r\sin\theta}\partial_\theta\big(A_\theta\sin\theta\big) + \frac{1}{r\sin\theta}\partial_\phi A_\phi \tag{15}$$

$$\begin{aligned}\nabla\times\boldsymbol{A} = \operatorname{rot}\boldsymbol{A} &= \frac{\boldsymbol{e}_r}{r\sin\theta}\big(\partial_\theta(A_\phi\sin\theta) - \partial_\phi A_\theta\big) + \frac{\boldsymbol{e}_\theta}{r}\Big(\frac{1}{\sin\theta}\partial_\phi A_r - \partial_r(rA_\phi)\Big)\\ &\quad + \frac{\boldsymbol{e}_\phi}{r}\big(\partial_r(rA_\theta) - \partial_\theta A_r\big)\end{aligned} \tag{16}$$

$$\Delta f = \frac{1}{r}\partial_{rr}(rf) + \frac{1}{r^2\sin\theta}\partial_\theta\big(\sin\theta\partial_\theta f\big) + \frac{1}{r^2\sin^2\theta}\partial_{\phi\phi} f \tag{17}$$

$$\begin{aligned}\Delta\boldsymbol{A} &= \Big(\Delta A_r - \frac{2}{r^2}A_r - \frac{2}{r^2\sin\theta}\big[\partial_\theta(\sin\theta A_\theta) + \partial_\phi A_\phi\big]\Big)\boldsymbol{e}_r\\ &\quad + \Big(\Delta A_\theta + \frac{2}{r^2}\partial_\theta A_r - \frac{A_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\partial_\phi A_\phi\Big)\boldsymbol{e}_\theta\\ &\quad + \Big(\Delta A_\phi + \frac{2}{r^2\sin\theta}\partial_\phi A_r + \frac{2\cos\theta}{r^2\sin^2\theta}\partial_\phi A_\theta - \frac{A_\phi}{r^2\sin^2\theta}\Big)\boldsymbol{e}_\phi\end{aligned} \tag{18}$$

$$\begin{aligned}\big(\boldsymbol{B}\cdot\nabla\big)\boldsymbol{A} &= \boldsymbol{e}_r\Big(\boldsymbol{B}\cdot\nabla A_r - \frac{B_\theta A_\theta}{r} - \frac{B_\phi A_\phi}{r}\Big) + \boldsymbol{e}_\theta\Big(\boldsymbol{B}\cdot\nabla A_\theta - \frac{B_\phi A_\phi}{r}\cot\theta + \frac{B_\theta A_r}{r}\Big)\\ &\quad + \boldsymbol{e}_\phi\Big(\boldsymbol{B}\cdot\nabla A_\phi + \frac{B_\phi A_r}{r} + \frac{B_\phi A_\theta}{r}\cot\theta\Big)\end{aligned} \tag{19}$$

Bessel functions $u(r) = J_n(r)$ and $u(r) = Y_n(r)$ are solutions to the ODE

$$r^2u'' + ru' + \big(r^2 - n^2\big)u = 0. \tag{20}$$

Both $J_n(r)$ and $Y_n(r) \to 0$ as $r \to \infty$; $J_n(0) = \delta_{n0}$, and $|Y_n(r)| \to \infty$ as $r \to 0$.