EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2020	MATH3111-WE01

Title: Quantum Mechanics III

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. Show your working and explain your reasoning.

Revision:	

ED01/2020

Q1 **1.1** Consider the observables $\hat{A}$ and $\hat{B}$ acting on a two dimensional Hilbert space which have the matrix forms

$$\hat{A} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \qquad \hat{B} = \begin{pmatrix} a & 3 \\ 3 & b \end{pmatrix},$$

with a and b real numbers.

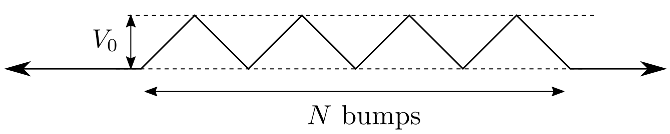
(i) Find the condition that a and b have to satisfy so that $\hat{A}$ and $\hat{B}$ admit a common eigenbasis.

(ii) What are the possible outcomes in a measurement of $\hat{A}$?

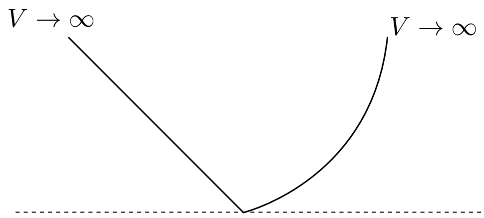
(iii) The observable $\hat{A}$ is measured and the result of the experiment is the highest value possible. If $\hat{B}$ is measured right after, what would be the possible outcomes of the measurement and with what probability? Express your answer in terms of the constant a.

1.2 The potentials for two different one-dimensional quantum systems are shown in the figure below:

1)



2)



A quantum particle propagates in one dimension under the influence of these potentials. Answer the following questions:

(i) Is the energy eigen spectrum of the particle in the first potential discrete or not? If only part of the spectrum is discrete, explain when this happens.

(ii) What happens with the spectrum of the particle in the first potential in the limit when N, the number of bumps, goes to infinity, i.e. when the potential in which the particle propagates becomes periodic?

(iii) Is the energy eigen spectrum of the particle in the second potential discrete or not? If only part of the spectrum is discrete, explain when this happens.

(iv) For the particle in the second potential, sketch the wave function for a highly excited particle of energy E. Please indicate where the probability to find the particle is the smallest and where is it the largest and write down the equations which give the points x_1 and x_2 where the wave function changes its behaviour.

Q2 Suppose that two linear operators $\hat{A}$ and $\hat{B}$ satisfy the commutation relation

$$[\hat{A}, \hat{B}] = 1 + \hat{A}^2 \,.$$

2.1 Show that

$$\left[\hat{A}^n, \hat{B}\right] = n\left(\hat{A}^{n-1} + \hat{A}^{n+1}\right) \,,$$

for all positive integers n.

2.2 Suppose that $f(x)$ is a function admitting a Taylor expansion around $x = 0$. By using the result of **2.1**, evaluate the commutator

$$\left[f(\hat{A}), \hat{B}\right] \,,$$

in terms of $\hat{A}$ and the function $f(x)$ (or its derivatives) evaluated at $x = \hat{A}$.

2.3 Suppose that $\mid \lambda\rangle$ is an eigenvector of $\hat{B}$ with eigenvalue λ. Construct an ordinary differential equation that the function $f_q(x)$ must satisfy in order to guarantee that $f_q(\hat{A}) \mid \lambda\rangle$ is also an eigenvector of $\hat{B}$ with eigenvalue $\lambda + q$.

2.4 Solve the ordinary differential equation that you found in **2.3** in order to find the most general form of $f_q(x)$.

2.5 Consider the case where $f_q(0) = 1$. What condition should be imposed on $\hat{A}$ in order for the operator $f_q(\hat{A})$ to be unitary when q is real?

Q3 The Hamiltonian of a Quantum Mechanical system takes the form

$$\hat{H} = w((\hat{a}^\dagger \hat{a})^3 + \hat{a}^\dagger \hat{a}) \,,$$

with $[\hat{a}, \hat{a}^\dagger] = 1$ and $w > 0$. A complete set of Hamiltonian eigen states $\{|n\rangle \,, n \in \mathbb{N}\}$ is given by $|n\rangle = c_n \, (\hat{a}^\dagger)^n \, |0\rangle$, for some constants of normalisation c_n and $\hat{a}\,|0\rangle = 0$.

3.1 Use the form of the Hamiltonian eigen states $|\, n\rangle$ to find the corresponding eigenvalues E_n of the Hamiltonian.

3.2 Fix the normalisation constants c_n so that the eigen states $|n\rangle$ are normalised, provided that $|0\rangle$ is.

3.3 Write the most general state $|\,\psi_0\rangle$ which can only yield the results 0 and $2w$ in an energy measurement with equal probability.

3.4 Consider the state $|\,\psi_0\rangle$ that you found in the previous question as the initial state and write down the time evolved state $|\,\psi(t)\rangle$.

3.5 For any positive integer m we can define the operator $\hat{C}_m = (\hat{a}^\dagger)^m + z\,\hat{a}^m$, with z a complex number. For what value of z can $\hat{C}_m$ represent a physical observable? For that value of z, compute the expectation value $\langle\, \hat{C}_m \,\rangle$ when the system is in the time dependent state $|\,\psi(t)\rangle$ that you determined in **3.4**.

Q4 A particle is placed in a spherically symmetric potential given by

$$V(r) = \begin{cases} 0\,, & 0 < r \leq R_1\,, \\ V_0\,, & R_1 < r \leq R_2\,, \\ \infty\,, & r > R_2\,, \end{cases}$$

where $r^2 = x^2 + y^2 + z^2$ and V_0 is a positive constant.

4.1 Write down the ansatz for the wave function $\psi(r,\theta,\varphi)$ which is adapted to the symmetry of the problem. By separation of variables, write down the Schrödinger equation as a system of two equations, one for the radial part and one for the angular part. In order to do so, you will need the expression for the Laplacian in spherical coordinates

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) - \frac{1}{r^2}\hat{L}^2\,,$$

where $\hat{L}$ is the coordinate representation of the angular momentum in three dimensions.

4.2 Given that the potential is piece-wise constant, your wave function will be different in the three regions ($0 < r \leq R_1,\ R_1 < r \leq R_2,\ r > R_2$). Write down the boundary and gluing conditions which the wave function has to satisfy in these three regions.

4.3 Simplify the radial part of the Schrödinger equation by using the following ansatz for the radial function, $R(r)$

$$R(r) = u(r)/r\,.$$

4.4 Focus on the ground state of the system, and by solving the radial part of the Schrödinger equation find the general form of the wave function. Explain how the equation which fixes the energy of the ground state follows from the boundary and gluing conditions. You do not need to derive this equation. Assume also that the energy of the ground state is smaller than V_0 .

Q5 The simple harmonic oscillator (SHO) in four dimensions (four spatial directions (x, y, z, w) and time) has Hamiltonian given by

$$\hat{H}_0 = \frac{1}{2}(\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2 + \hat{p}_w^2) + \frac{\kappa}{2}(\hat{x}^2 + \hat{y}^2 + \hat{z}^2 + \hat{w}^2)\,,$$

where $\kappa > 0$ and is perturbed by the Hamiltonian

$$\hat{H}' = \frac{\alpha}{2}\hat{x}^2 + \frac{\beta}{2}\hat{y}^2\,,$$

where $\alpha > 0, \beta > 0$ are small constants.

5.1 Write down the ground state and all *first* energy eigen states of the unperturbed system ($\alpha = \beta = 0$). Write down the generic n-th energy eigen state in the spectrum (you do not have to normalise it). What is the energy of this n-th energy eigen state? What is the degeneracy of this n-th eigen state?

5.2 Apply first-order perturbation theory to compute the shift to the energy of the ground state, when $\alpha \neq 0, \beta \neq 0$.

5.3 Apply first-order perturbation theory to compute the corrections to the energy of the first excited states, when $\alpha \neq 0, \beta \neq 0$. What is the effect of the perturbation $\hat{H}'$ on the degeneracy of the spectrum of the unperturbed Hamiltonian?

5.4 By treating $H_0 + H'$ as the new Hamiltonian, write down the exact expression of the energy of the first excited state of this new Hamiltonian. By first excited state we mean any state for which the total number of excitations is one.

Hint: When computing shifts in energy you can use the following formula which holds for creation and annihilation operators of the simple harmonic oscillator $\hat{x} = A(\hat{a}_x + \hat{a}_x^\dagger)$ where $A^2 = \hbar/(2\sqrt{\kappa})$.