Durham University

EXAMINATION PAPER

Examination Session:	**Year:**	**Exam Code:**
May/June	2020	MATH3141-WE01

Title:
Operations Research III

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. Show your working and explain your reasoning.

Revision:	

ED01/2020

Q1 **1.1** Solve the following linear program by using the two phase method, and state all feasible values of x_1, x_2, and x_3 for which the optimal value is attained.

$$\begin{array}{rrcl} \max & x_1 - 2x_2 + x_3 & & \\ \text{subject to} & -x_1 + x_2 & \le & -2 \\ & x_1 + x_2 + x_3 & \le & 3 \\ & x_1 & \ge & 0 \\ & x_2 & \ge & 0 \\ & x_3 & \ge & 0 \end{array}$$

1.2 A generous member of the public has £17 to donate to charity, and has identified three possible charities that support the same cause.

- Charity 1 accepts donations in multiples of £8. For every £8 donation, £5.50 ends up going to the cause.
- Charity 2 accepts donations in multiples of £4. The first £4 donated results in £2 going to the cause, due to a one-off administration fee. For every further £4 donated, £3 ends up going to the cause.
- Charity 3 accepts donations in multiples of £3. For every £3 donation, £2 ends up going to the cause.

Formulate the above as a dynamic programming problem, and solve it, to determine how the £17 should be spent so that the maximum amount of money ends up going to the cause.

Q2 Consider a case where someone has solved the following linear programming problem:

$$\max\ 3x_1+2x_2+3x_3+3x_4 \tag{1a}$$
$$\text{subject to } 4x_1+\ x_2+\ x_3+\ x_4 \le 3 \tag{1b}$$
$$x_1+\ x_2+2x_3+3x_4 \le 4 \tag{1c}$$
$$3x_1+2x_2+2x_3+3x_4 \le 10 \tag{1d}$$

and subject to all $x_i \geq 0$ for $i \in \{1,2,3,4\}$. The initial simplex table for this problem is

$$\begin{array}{c|ccccccc|c} T_0 & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & \\ \hline z & -3 & -2 & -3 & -3 & 0 & 0 & 0 & 0 \\ \hline s_1 & 4 & 1 & 1 & 1 & 1 & 0 & 0 & 3 \\ s_2 & 1 & 1 & 2 & 3 & 0 & 1 & 0 & 4 \\ s_3 & 3 & 2 & 2 & 3 & 0 & 0 & 1 & 10 \end{array} \tag{2}$$

and the final simplex table is

$$\begin{array}{c|ccccccc|c} T_* & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & \\ \hline z & 2 & 0 & 0 & 1 & 1 & 1 & 0 & 7 \\ \hline x_2 & 7 & 1 & 0 & -1 & 2 & -1 & 0 & 2 \\ x_3 & -3 & 0 & 1 & 2 & -1 & 1 & 0 & 1 \\ s_3 & -5 & 0 & 0 & 1 & -2 & 0 & 1 & 4 \end{array} \tag{3}$$

Now consider a similar linear programming problem, with modifications as indicated in bold:

$$\max\ 3x_1+2x_2+3x_3+3x_4\boldsymbol{+4x_5} \tag{4a}$$
$$\text{subject to } 4x_1+\ x_2+\ x_3+\ x_4\boldsymbol{+\alpha x_5} \le 3\boldsymbol{+\alpha} \tag{4b}$$
$$x_1+\ x_2+2x_3+3x_4\boldsymbol{+\alpha x_5} \le 4\boldsymbol{+\alpha} \tag{4c}$$
$$3x_1+2x_2+2x_3+3x_4\boldsymbol{+\alpha x_5} \le 10\boldsymbol{+\alpha} \tag{4d}$$

subject to all $x_i \geq 0$ for $i \in \{1,2,3,4,5\}$. Here, α is an arbitrary fixed value in $\mathbb{R}$. For this modified problem, *use post-optimal analysis* to answer all of the following questions:

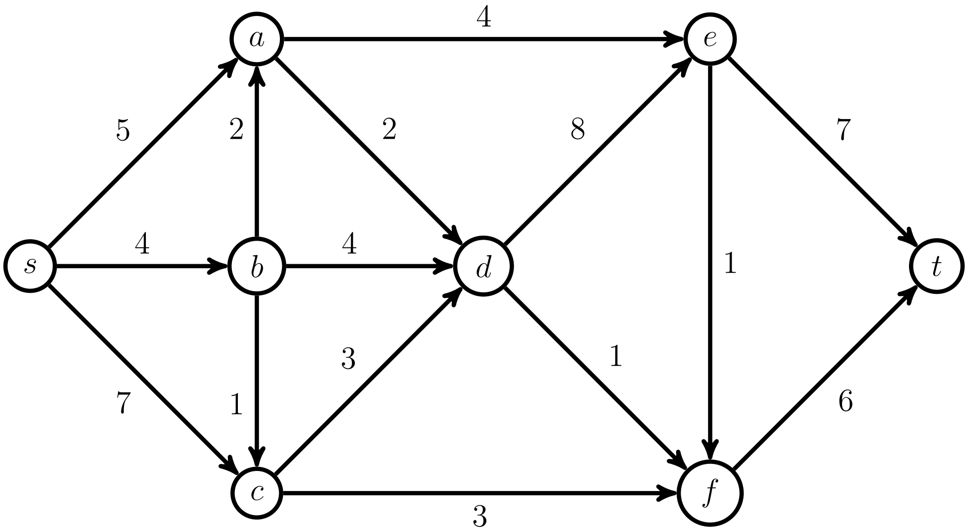
2.1 For what values of $\alpha \in \mathbb{R}$ does the basis $\{x_2, x_3, s_3\}$ give a basic feasible solution?

2.2 For what values of $\alpha \in \mathbb{R}$ does the basis $\{x_2, x_3, s_3\}$ give an optimal basic feasible solution?

2.3 In those cases where the basis $\{x_2, x_3, s_3\}$ gives an optimal basic feasible solution, what is the optimal value of the objective function as a function of α?

2.4 Find the optimal solution of the modified problem for $\alpha = 1$.

Q3 Consider a transportation system that can be represented by the following flow network, with capacities as labelled:



3.1 Apply the Ford-Fulkerson algorithm to find the maximum flow from the source s to the terminus t.

3.2 Find a cut separating s and t with minimal capacity.

3.3 Identify one arc that provides an opportunity for decreasing the maximal flow, and provide a theoretical argument as to why your chosen arc provides this opportunity. You may use, without proof, any theorems proved in the lectures, as long as you state them clearly.

3.4 Decrease the capacity of the arc that you just identified in question **3.3** by 1, and find the new maximal flow as well as a new cut between s and t with minimal capacity.

Q4 **4.1** A stochastic optimization problem can be divided into N stages, with possible states $s \in \mathcal{S}_n$ and actions $x \in \mathcal{A}_n$ at stage $n \leq N$. Let $\mathcal{S}_{N+1}$ denote the final set of states. The probability of going to state $j \in \mathcal{S}_{n+1}$ from state $i \in \mathcal{S}_n$, when taking action $x \in \mathcal{A}_n$ at stage n, is denoted by $P_n(i, j; x)$.

Write down the backwards induction algorithm for stochastic dynamic programming, with the aim of maximizing the probability of an event $F \subset \mathcal{S}_{N+1}$. Define carefully any notation you introduce.

4.2 An investment banker has £1000 and has identified two possible investment schemes. Each accepts investments in multiples of £1000 at the beginning of the year and provides returns at the end of the same year. Scheme A provides a return of £2000 with probability p, and £0 with probability $(1-p)$. Scheme B provides a return of £2000 with probability q, and £1000 with probability $(1-q)$.

	annual return on $x\times$£1000 investment	probability
A	£0	$1-p$
	$x\times$£2000	p
B	$x\times$£1000	$1-q$
	$x\times$£2000	q

Assume that

$$0 < q < p < 1.$$

The banker can make a *single* investment at the beginning of each year, in any multiple of £1000 up to the amount they have at the time. Use the algorithm from question **4.1** to determine the optimal policy for maximizing the probability that at the end of the third year, the banker has at least £4000.

How does this change if the banker wants to maximize the probability of having at least £8000?

Q5 A baker produces a batch of bread each morning. The bread is either good or bad. If the bread is good, the baker will sell all of it and make £60 that day. If the bread is bad the baker will have to sell it at a heavily discounted price, and will only make £10.

Each morning, the baker must decide whether to use the same recipe as the previous day, or try a new recipe. It costs the baker £10 to try a new recipe.

When sticking with the same recipe, the transition probabilities from one day to the next are given by:

	bad	good
bad	0.9	0.1
good	0.1	0.9

When changing recipe they are given by:

	bad	good
bad	0.6	0.4
good	0.6	0.4

Calculate

$$r(\text{good}, \text{change})\,;\; r(\text{good}, \text{stick})\,;\; r(\text{bad}, \text{change}) \text{ and } r(\text{bad}, \text{stick})$$

where $r(\alpha, \beta)$ is the expected £ profit in a day when action β is taken, and the previous day's batch was of quality α.

The baker's current policy is to always stick to the same recipe, but they would like to maximize their expected long-run average reward. Perform two steps of the policy improvement algorithm for this goal. What is the resulting policy? How do you know that this is the optimal policy?