Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2020	MATH3171-WE01

Title:
Mathematical Biology

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. Show your working and explain your reasoning.

Revision:	

ED01/2020

Q1 Reaction diffusion modelling and similarity solutions Consider a bacterial population density c living on a growing substrate V. The density is modelled using the reaction diffusion equation

$$\frac{\partial c}{\partial t} = \nabla^2 c + \nabla \cdot (c\mathbf{v}),$$

with $\mathbf{v}$ an advective velocity. We assume V is $\mathbb{R}^2$ and equipped with polar coordinates (r, θ) (with unit basis vectors $(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}})$).

1.1 (i) The advective velocity is assumed to take the following form:

$$\mathbf{v} = v_r \hat{\mathbf{r}} = \frac{c_0 - c}{r^2}\hat{\mathbf{r}}.$$

Describe the physical interpretation of this advective velocity. Focus on how it relates substrate growth to, (a) the bacterial population, and (b) the radial coordinate.

(ii) The bacterial density is assumed to maintain radial symmetry. The equilibrium equation for the model described in part **1.1**(i) is given by the following equation:

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\frac{\mathrm{d}c}{\mathrm{d}r}\right) - \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r v_r c) = 0.$$

Describe the behaviour of c as $r \to \infty$, subject to the constraint that $\mathrm{d}c/\mathrm{d}r = 0$ when $c = 0$ for all r.

1.2 (i) Consider the following Korteweg-de Vries type equation for a scalar density $u(x, t)$

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + a\frac{\partial^3 u}{\partial x^3} = 0, \tag{1}$$

where $x \in \mathbb{R}$, $t > 0$ and $a \in \mathbb{R}$ is a constant.
Assume a similarity solution v which relates to u as

$$u(x, t) = \frac{1}{t^\alpha}v(y), \quad y = x/t^\beta.$$

Find values of α and β such that (1) reduces to the following O.D.E for v.

$$-\alpha v - y\beta\frac{\mathrm{d}v}{\mathrm{d}y} - 6v\frac{\mathrm{d}v}{\mathrm{d}y} + a\frac{\mathrm{d}^3 v}{\mathrm{d}y^3} = 0.$$

(ii) Assume that $a = 0$. Consider a one parameter similarity reduction of (1),

$$u(x, t) = \frac{1}{t^\alpha}v(y), \quad y = x/t^\alpha.$$

Show, by integration, that in this case the solutions to (1) take the form:

$$u(x, t) = \frac{1}{12t^{1/2}}\left[-\frac{x}{t^{1/2}} \pm \sqrt{\frac{x^2}{t} - C}\right],$$

where C is a constant.

Q2 Planar elastic rods It can be shown that the equilibrium equations of a slender elastic rod, parameterised by its arclength $s \in [0, L]$, satisfies the following pair of ordinary differential equations:

$$\begin{aligned} A\frac{\mathrm{d}^2 u}{\mathrm{d}s^2} - nu + f &= 0, \\ \frac{\mathrm{d}n}{\mathrm{d}s} + A\frac{\mathrm{d}u}{\mathrm{d}s}u &= 0, \end{aligned} \tag{2}$$

where $u(s)$ is the curvature of the rods's axis curve, $n(s)$ is the axial force in the rod's cross section, f an externally applied body force density and A its constant bending stiffness.

2.1 Reduce (2) to a single ordinary differential equation for u.

2.2 Consider a body force density in the form $f = ku$, what does this represent physically? Remember that f represents the magnitude of a vector.

2.3 Assume the rod is clamped at its ends, $u(0) = u(L) = 0$, and subject to an applied load $N > 0$. Assume the body force takes the form $f = ku$. Assess the (static) stability of the equilibrium when the rod's axis is a straight line. Interpret this result by comparison to the Euler buckling criterion which is the case $k = 0$.

2.4 Now consider a second model for the equilibrium of a planar, unstretchable, slender elastic rod, $\mathbf{r}(s) : [0, L] \to \mathbb{R}^3$, where s is the body's arclength. The coordinates of $\mathbb{R}^3$ take the form (x, y, z) with unit vectors $\hat{\mathbf{x}}, \hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ respectively. The rod's unit tangent vector $\mathbf{d}_3$ takes the form

$$\mathbf{d}_3 = (\cos\theta(s), 0, \sin\theta(s)).$$

The equilibrium equation of this rod takes the form

$$B\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} - b\frac{\mathrm{d}\theta}{\mathrm{d}s} = \sin(n\pi s/L), \tag{3}$$

where $n > 0$ is an integer, and $B, b > 0$ are constants. Solve (3) subject to the boundary conditions that the rod points along $\hat{\mathbf{x}}$ at $s = 0$ and it has no curvature at $s = L$. Describe (roughly) the shape of the rod $\mathbf{r}(s)$ given by the solution $\theta(s)$ in the limit $L \to \infty$. Is this physically realistic?

Q3 Turing Analysis Consider the following reaction diffusion system for the interaction of a morphogen chemical density u and a biocatalyst density v,

$$\frac{\partial u}{\partial t} = \nabla^2 u + \gamma F(u,v),$$
$$\frac{\partial v}{\partial t} = D\nabla^2 v + \gamma G(u,v),$$

where the constant D is the ratio of diffusion constants of v and u, and γ is also a positive constant. We assume the domain V is Cartesian, 2-dimensional, and both u and v are subject to fluxless boundary conditions. For this system the Turing conditions for pattern formation take the following form

$$\begin{aligned} &F_u + G_v < 0, \quad F_uG_v - F_vG_u > 0, \\ &G_v + DF_u > 0, \quad (G_v + DF_u)^2 - 4D(F_uG_v - F_vG_u) \geq 0. \end{aligned} \tag{4}$$

Here we have used the notation

$$F_u = \left.\frac{\partial F}{\partial u}\right|_{u=u_0,v=v_0}, \quad F_v = \left.\frac{\partial F}{\partial v}\right|_{u=u_0,v=v_0},$$
$$G_u = \left.\frac{\partial G}{\partial u}\right|_{u=u_0,v=v_0}, \quad G_v = \left.\frac{\partial G}{\partial v}\right|_{u=u_0,v=v_0}.$$

3.1 State the condition on the densities u and v required for them to be physically admissible.

3.2 Consider the following reaction functions,

$$F(u,v) = a - u - h(u,v), \quad G(u,v) = b - v - h(u,v),$$

where $a, b > 0$ are constants and $h(u,v)$ is a smooth non-linear function of **both** u and v.

Describe possible physical interpretations of the various terms in these reaction functions. Focus on whether they are inter or intra species interactions. If they are inter species interactions comment on whether they are cooperative or competitive.

3.3 Demonstrate that the first three Turing conditions are satisfied if the following inequalities for h are satisfied

$$(D+1) + Dh_u + h_v < 0, \text{and } 1 + h_u + h_v > 0,$$

where

$$h_u = \left.\frac{\partial h}{\partial u}\right|_{u=u_0,v=v_0}, \quad h_v = \left.\frac{\partial h}{\partial v}\right|_{u=u_0,v=v_0}.$$

3.4 Consider the following two functions

$$h(u,v) = \gamma uv, \quad h(u,v) = u^2 - \gamma v^2 + v,$$

where $\gamma > 0$ is a constant. Show that if $a = b$ and $\gamma < 1/u_0$ then neither case can satisfy the Turing conditions.

3.5 Assume $h(u, v)$ takes the following form:

$$h(u, v) = \gamma u^2 - v^2.$$

Show that if $a = b$ there exists a physically valid equilibrium (u_0, v_0) such that u_0 can take on any value on $(0, \infty)$. Further show (if $a = b$) that one can satisfy all the Turing conditions (4) if $D \in (0, 1)$ and state the range of u_0 values for which this is the case.

Q4 Species interaction Consider the following temporal model of the interaction of two animal populations $\hat{u}(\hat{t})$ and $\hat{v}(\hat{t})$, $\hat{t} \in (0, \infty]$,

$$\begin{aligned} \frac{\mathrm{d}\hat{u}}{\mathrm{d}\hat{t}} &= a\hat{u}\mathrm{e}^{-\hat{u}} + b\hat{u}\hat{v}, \\ \frac{\mathrm{d}\hat{v}}{\mathrm{d}\hat{t}} &= a\hat{v}\left(1 - \frac{\hat{v}}{c}\right) + d\hat{u}\hat{v}, \end{aligned} \tag{5}$$

where $a, b, c > 0$ are real positive constants and d a real constant.

In the following question we will use the Exponential integral $\mathrm{E}_i(x)$ whose definition takes the from

$$\mathrm{E}_i(x) = \int_{-\infty}^{x} \frac{\mathrm{e}^y}{y}\mathrm{d}y.$$

4.1 Show that we can introduce scaled variables, u, v, t such that (5) can be written as

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}t} &= u\mathrm{e}^{-\delta u} + \gamma_1 uv, \\ \frac{\mathrm{d}v}{\mathrm{d}t} &= v(1 - v) + \gamma_2 uv \end{aligned} \tag{6}$$

where $\gamma_1, \delta > 0$ are constants and γ_2 a real constant.

4.2 Assume $v(t) = 0\,\forall t$. Solve (6) for u, with $u(0) = u_c > 0$ and $\delta = 1$. Is the growth monotonic? Justify your reasoning.

4.3 Consider the following growth model

$$\frac{\mathrm{d}u}{\mathrm{d}t} = g(u),$$

with the specific cases

(i) $g(u) = u$,

(ii) $g(u) = u(1 - u)$,

(iii) $g(u) = u\exp(-u)$.

Compare and contrast the three models. Hint: you do not need to solve any of these models to answer the question and can freely state any facts obtained in class.

4.4 Find all **physically permissible** equilibria of the system (6) and analyse their linear stability.

Q5 Non-Euclidean reaction diffusion Consider the following generic reaction diffusion system for two scalar densities u and v

$$\begin{aligned}\frac{\partial u}{\partial t} &= \nabla^2 u + F(u,v),\\ \frac{\partial v}{\partial t} &= D\nabla^2 v + G(u,v),\end{aligned} \tag{7}$$

where $D > 0$ is a constant diffusion coefficient. The system is assumed to have a non-zero homogeneous equilibrium $u_0 > 0, v_0 > 0$. We assume the densities interact in a two dimensional non-Euclidean domain with coordinates (r, s). In an arbitrary orthogonal coordinate system (r, s) the Laplacian function takes the form

$$\nabla^2\psi = \frac{1}{h_s h_r}\left[\frac{\partial}{\partial s}\left(\frac{h_r}{h_s}\frac{\partial \psi}{\partial s}\right) + \frac{\partial}{\partial r}\left(\frac{h_s}{h_r}\frac{\partial \psi}{\partial r}\right)\right]$$

where $h_r(r, s)$ and $h_s(r, s)$ are functions specifying the stretching of the coordinate system along the r and s directions respectively.

5.1 Assume solutions to (7) in the form $u(r, s, t) = u_0 + \epsilon u_1(r, s, t)$ and $v(r, s, t) = v_0 + \epsilon v_1(r, s, t)$. State the linearised form of the system.

5.2 Assume the solutions to the linearised system for both u_1 and v_1 are proportional to $\psi(r, s)\mathrm{e}^{\lambda t}$ where the spatial function ψ satisfies

$$\nabla^2\psi + k^2\psi = 0. \tag{8}$$

The solutions of (8) give the functions which dictate the allowed patterns on this domain. Consider a domain for which $s \in [0, 2\pi)$, the coordinate system is periodic in s, and $r \in [R_1, R_2]$ where $R_2 > R_1 > 0$; this is a distorted annular domain. Assume further that $h_s = h_r$. Show that the patterns can be defined by functions $f(r)$ which satisfy

$$\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + (h_s^2 k^2 - n^2)f = 0, \tag{9}$$

for integers $n \geq 0$.

5.3 Consider the $n = 0$ mode. Solve (9) for $f(r)$ on the assumption that $h_s = 1/r$. Only include real solutions.You should assume solutions in the form $f(r) = r^{\lambda}$. Hint: you should encounter imaginary solutions which can be written in terms of trigonometric functions.

5.4 Assume patterns form via the Turing instability in the particular system considered in parts **5.2** and **5.3** with **no flux boundary conditions**, that is to say **inhomogeneous** patterns form. Further assume all patterns are observed to be homogeneous in the s coordinate direction. Consider cases for which (i) $R_2 \ll 1$ and (ii) $R_1 \gg 1$. Assume further the patterns only form if $u_1 > 1$. Describe the form each individual pattern mode takes in both cases.