

EXAMINATION PAPER

Examination Session: May/June

2020

Year:

Exam Code:

MATH3231-WE01

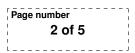
Title:

Solitons III

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks.
	Please start each question on a new page. Please write your CIS username at the top of each page.
	Show your working and explain your reasoning.

Revision:



Q1 1.1 The function u(x,t) satisfies a linear partial differential equation of the type

$$\sum_{m,n} a_{m,n} \frac{\partial^{m+n} u}{\partial^m x \partial^n t} = 0 ,$$

where m, n are non-negative integers and only a finite number of the constants a_{mn} are non-vanishing.

- (i) Define the terms dispersion relation, phase velocity and group velocity for this type of equation.
- (ii) Find the most general dispersion relation that ensures that the group velocity is proportional to the phase velocity.
- (iii) Find the dispersion relation for the equation

$$\frac{\partial^3 u}{\partial^3 t} + 3 \frac{\partial^3 u}{\partial^2 t \partial x} - \frac{\partial^3 u}{\partial t \partial^2 x} - 3 \frac{\partial^3 u}{\partial^3 x} = 0 \ .$$

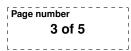
1.2 Construct a smooth travelling wave solution of Burgers' equation

$$u_t + uu_x - u_{xx} = 0$$

subject to the boundary conditions

 $\begin{array}{lll} u \to A \ , & u_x \to 0 \ \mbox{ as } \ x \to -\infty \ , \\ u \to B \ , & u_x \to 0 \ \ \mbox{as } \ x \to +\infty \ , \end{array}$

where A, B are real constants with A > B > 0.



Q2 It is known that

$$Q_1 = \int_{-\infty}^{+\infty} dx \ u \ , \qquad Q_2 = \int_{-\infty}^{+\infty} dx \ u^2 \ , \qquad Q_3 = \int_{-\infty}^{+\infty} dx \ \left(u^3 - \frac{1}{2} u_x^2 \right)$$

and

$$Q_* = \int_{-\infty}^{+\infty} dx \ (xu - 3tu^2)$$

are conserved charges for the KdV equation $u_t + 6uu_x + u_{xxx} = 0$ subject to boundary conditions that u decays exponentially fast as $|x| \to \infty$.

2.1 Using the integral

$$\int_{-\infty}^{+\infty} dx \operatorname{sech}^{2n}(x) = \frac{2^{2n-1}(n-1)!^2}{(2n-1)!} ,$$

evaluate Q_1 , Q_2 and Q_3 for the field configuration $u = a \operatorname{sech}^2(bx)$, where a and b are constants.

According to the KdV equation, the initial condition

$$u(x,0) = N(N+1)\operatorname{sech}^{2}(x)$$
, (1)

where N is an integer, is known to evolve at late times into the sum of N wellseparated solitons with velocities $4k^2$, $k = 1 \dots N$:

$$u(x,t) \approx \sum_{k=1}^{N} 2k^2 \operatorname{sech}^2 \left[k(x - x_k - 4k^2 t) \right], \qquad t \to +\infty , \qquad (2)$$

where x_1, x_2, \ldots, x_N are constants which are determined by the initial condition.

- **2.2** Use the conservation of Q_1 , Q_2 and Q_3 for the solution of the KdV equation with initial condition (1) to deduce formulae for the sums of the first N integers, the first N cubes, and the first N fifth powers.
- **2.3** Use the conservation of Q_* for the solution of the KdV equation with initial condition (1) to deduce a linear relation among the parameters x_k appearing in the late time asymptotics (2). For N = 3, is the set of constants $(x_1, x_2, x_3) = (-10, 0, 10)$ possible?

 $\mathbf{Q3}$ Consider the time-independent Schrödinger equation

$$-\psi''(x) + V(x)\psi(x) = E\psi(x)$$

with the potential

$$V(x) = \begin{cases} 0 , & x < 0 \\ 2 \operatorname{sech}^2 x , & x > 0 \end{cases}$$

3.1 If the corresponding solution is defined to be

$$\psi(x) = \begin{cases} \psi_{-}(x) , & x < 0\\ \psi_{+}(x) , & x > 0 \end{cases},$$

write down the conditions that ψ_{-} , ψ_{+} , ψ'_{-} and ψ'_{+} must obey at x = 0.

 $\mathbf{3.2}$ Given that

$$\psi_+(x) \propto (\tanh(x) - ik)e^{ikx}$$
,

for $E = k^2$, calculate the reflection coefficient R(k) and the transmission coefficient T(k) for the potential given above and verify that they satisfy the relation

$$|R(k)|^2 + |T(k)|^2 = 1$$
.

3.3 For which complex values of k does T(k) computed in part **3.2** have a pole? Which of these corresponds to a bound state? Write down the corresponding (unnormalized) bound state solution.



Q4 The operators L and M are defined to be

$$L = D^2 + u$$
$$M = -4D^3 - 3Du - 3uD$$

where u = u(x, t) and $D \equiv \frac{\partial}{\partial x}$.

4.1 Show that the equation

$$\left[\frac{\partial}{\partial t} - M, L\right]\psi(x, t) = 0$$

is obeyed if u(x,t) satisfies the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \; .$$

4.2 Assuming that $\psi(x,t)$ does not vanish identically, show that the equation

$$\left[i\frac{\partial}{\partial t}+L,-D+M+w(x,t)\right]\psi(x,t)=0$$

implies that u(x,t) satisfies the following version of the Boussinesq equation,

$$(3u^2 + u + u_{xx})_{xx} - 3u_{tt} = 0 ,$$

provided that w_x is related to u_t in a way that you should determine.

Q5 5.1 The two functions u(x,t) and v(x,t) are related by the pair of equations

$$v_x + vu_x = e^u , \qquad v_t - vu_t = -v^2 e^u .$$

Show that this defines a Bäcklund transform between an equation for u and one for v.

5.2 A Lax pair of matrices L and M is given by

$$L = \begin{bmatrix} p_1 & b_1 & 0 \\ b_1 & p_2 & b_2 \\ 0 & b_2 & p_3 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & b_1 & 0 \\ -b_1 & 0 & b_2 \\ 0 & -b_2 & 0 \end{bmatrix},$$

where $p_i = \dot{q}_i$ and $b_i = \exp[c(q_i - q_{i+1})]$ for some constant c. Use the Lax equation $\dot{L} + [L, M] = 0$ to find the constant c and obtain equations of motion in the form $\ddot{q}_i = f_i(\{q_j\})$. [Notation: in this question each dot denotes a time derivative.]