Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2020	MATH3231-WE01

Title:
Solitons III

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. Show your working and explain your reasoning.

Revision:	

ED01/2020

Q1 **1.1** The function $u(x,t)$ satisfies a linear partial differential equation of the type

$$\sum_{m,n} a_{m,n} \frac{\partial^{m+n} u}{\partial^m x \partial^n t} = 0 \ ,$$

where m, n are non-negative integers and only a finite number of the constants a_{mn} are non-vanishing.

(i) Define the terms dispersion relation, phase velocity and group velocity for this type of equation.

(ii) Find the most general dispersion relation that ensures that the group velocity is proportional to the phase velocity.

(iii) Find the dispersion relation for the equation

$$\frac{\partial^3 u}{\partial^3 t} + 3\frac{\partial^3 u}{\partial^2 t \partial x} - \frac{\partial^3 u}{\partial t \partial^2 x} - 3\frac{\partial^3 u}{\partial^3 x} = 0 \ .$$

1.2 Construct a smooth travelling wave solution of Burgers' equation

$$u_t + u u_x - u_{xx} = 0$$

subject to the boundary conditions

$$\begin{aligned} u \to A \ , \quad u_x \to 0 \quad \text{as} \quad x \to -\infty \ , \\ u \to B \ , \quad u_x \to 0 \quad \text{as} \quad x \to +\infty \ , \end{aligned}$$

where A, B are real constants with $A > B > 0$.

Q2 It is known that

$$Q_1 = \int_{-\infty}^{+\infty} dx\ u\ , \qquad Q_2 = \int_{-\infty}^{+\infty} dx\ u^2\ , \qquad Q_3 = \int_{-\infty}^{+\infty} dx\ \left(u^3 - \frac{1}{2}u_x^2\right)$$

and

$$Q_* = \int_{-\infty}^{+\infty} dx\ (xu - 3tu^2)$$

are conserved charges for the KdV equation $u_t + 6uu_x + u_{xxx} = 0$ subject to boundary conditions that u decays exponentially fast as $|x| \to \infty$.

2.1 Using the integral

$$\int_{-\infty}^{+\infty} dx\ \mathrm{sech}^{2n}(x) = \frac{2^{2n-1}(n-1)!^2}{(2n-1)!}\ ,$$

evaluate Q_1, Q_2 and Q_3 for the field configuration $u = a\,\mathrm{sech}^2(bx)$, where a and b are constants.

According to the KdV equation, the initial condition

$$u(x,0) = N(N+1)\,\mathrm{sech}^2(x)\ , \qquad (1)$$

where N is an integer, is known to evolve at late times into the sum of N well-separated solitons with velocities $4k^2$, $k = 1 \dots N$:

$$u(x,t) \approx \sum_{k=1}^{N} 2k^2\,\mathrm{sech}^2\big[k(x - x_k - 4k^2 t)\big]\ , \qquad t \to +\infty\ , \qquad (2)$$

where $x_1, x_2, \dots, x_N$ are constants which are determined by the initial condition.

2.2 Use the conservation of Q_1, Q_2 and Q_3 for the solution of the KdV equation with initial condition (1) to deduce formulae for the sums of the first N integers, the first N cubes, and the first N fifth powers.

2.3 Use the conservation of Q_* for the solution of the KdV equation with initial condition (1) to deduce a linear relation among the parameters x_k appearing in the late time asymptotics (2). For $N = 3$, is the set of constants $(x_1, x_2, x_3) = (-10, 0, 10)$ possible?

Q3 Consider the time-independent Schrödinger equation

$$-\psi''(x) + V(x)\psi(x) = E\psi(x)$$

with the potential

$$V(x) = \begin{cases} 0 \ , & x < 0 \\ 2\ \mathrm{sech}^2 x \ , & x > 0 \end{cases} .$$

3.1 If the corresponding solution is defined to be

$$\psi(x) = \begin{cases} \psi_-(x) \ , & x < 0 \\ \psi_+(x) \ , & x > 0 \end{cases} ,$$

write down the conditions that ψ_-, ψ_+, ψ'_- and ψ'_+ must obey at $x = 0$.

3.2 Given that

$$\psi_+(x) \propto (\tanh(x) - ik)e^{ikx} ,$$

for $E = k^2$, calculate the reflection coefficient $R(k)$ and the transmission coefficient $T(k)$ for the potential given above and verify that they satisfy the relation

$$|R(k)|^2 + |T(k)|^2 = 1 .$$

3.3 For which complex values of k does $T(k)$ computed in part **3.2** have a pole? Which of these corresponds to a bound state? Write down the corresponding (unnormalized) bound state solution.

Q4 The operators L and M are defined to be

$$L = D^2 + u$$
$$M = -4D^3 - 3Du - 3uD$$

where $u = u(x,t)$ and $D \equiv \frac{\partial}{\partial x}$.

4.1 Show that the equation

$$\left[\frac{\partial}{\partial t} - M, L\right] \psi(x,t) = 0$$

is obeyed if $u(x,t)$ satisfies the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \ .$$

4.2 Assuming that $\psi(x,t)$ does not vanish identically, show that the equation

$$\left[i\frac{\partial}{\partial t} + L, -D + M + w(x,t)\right] \psi(x,t) = 0$$

implies that $u(x,t)$ satisfies the following version of the Boussinesq equation,

$$(3u^2 + u + u_{xx})_{xx} - 3u_{tt} = 0 \ ,$$

provided that w_x is related to u_t in a way that you should determine.

Q5 **5.1** The two functions $u(x,t)$ and $v(x,t)$ are related by the pair of equations

$$v_x + vu_x = e^u \ , \qquad v_t - vu_t = -v^2 e^u \ .$$

Show that this defines a Bäcklund transform between an equation for u and one for v.

5.2 A Lax pair of matrices L and M is given by

$$L = \begin{bmatrix} p_1 & b_1 & 0 \\ b_1 & p_2 & b_2 \\ 0 & b_2 & p_3 \end{bmatrix} , \quad M = \begin{bmatrix} 0 & b_1 & 0 \\ -b_1 & 0 & b_2 \\ 0 & -b_2 & 0 \end{bmatrix} ,$$

where $p_i = \dot{q}_i$ and $b_i = \exp[c(q_i - q_{i+1})]$ for some constant c. Use the Lax equation $\dot{L} + [L, M] = 0$ to find the constant c and obtain equations of motion in the form $\ddot{q}_i = f_i(\{q_j\})$. [Notation: in this question each dot denotes a time derivative.]