

EXAMINATION PAPER

Examination Session: May/June

2020

Year:

Exam Code:

MATH3251-WE01

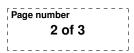
Title:

Stochastic Processes III

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks.
	Please start each question on a new page. Please write your CIS username at the top of each page.
	Show your working and explain your reasoning.

Revision:





- **Q1 1.1** Carefully define the Poisson process $(N(t))_{t\geq 0}$ with intensity $\lambda > 0$, and state its main properties.
 - **1.2** For fixed t > 0, write the formula for the distribution of N(t), i.e., for the values of probabilities P(N(t) = k) with integer $k \ge 0$. Prove your formula for the case k = 0.
 - **1.3** For fixed $0 < t_1 < t_2$ and an integer $n \ge 0$, find the conditional probability

$$\mathsf{P}(N(t_1) = k \mid N(t_2) = n)$$
, where $k = 0, 1, ..., n$.

Identify this distribution.

1.4 For integer n > 0, let T_n be the time of the *n*th event in $(N(t))_{t \ge 0}$. Find the probability density function of T_{n+1} given $T_n = s$, and deduce that the conditional joint density of T_1, T_2 given $N_t = 2$ satisfies

$$f_{T_1,T_2|N_t}(s_1, s_2 \mid 2) = \begin{cases} \frac{2}{t^2}, & \text{if } 0 < s_1 < s_2 \le t, \\ 0, & \text{otherwise.} \end{cases}$$

1.5 Let X_1 and X_2 be independent with common uniform distribution, $X_i \sim \mathcal{U}(0, t]$. Define (Y_1, Y_2) as the order statistic for (X_1, X_2) , namely, $Y_1 = \min(X_1, X_2)$ and $Y_2 = \max(Y_1, Y_2)$. Find the probability density function $f_{X_1, X_2}(x_1, x_2)$ and deduce that

$$f_{Y_1, Y_2}(y_1, y_2) = \begin{cases} \frac{2}{t^2} \,, & \text{if } 0 < y_1 < y_2 \le t \,, \\ 0 \,, & \text{otherwise} \,. \end{cases}$$

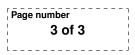
Deduce that $(T_1, T_2 | N_t = 2)$ from question **1.4** and (Y_1, Y_2) have the same distribution.

You should show all your working and justify your calculations with suitable explanation.

- **Q2** Let X_1, X_2, \ldots be independent identically distributed random variables with $\mathsf{P}(X_n = 1) = p = 1 \mathsf{P}(X_n = -1)$ for some $p \in [1/2, 1)$. Set $S_0 = 0$ and $S_n = \sum_{i=1}^n X_i$. For $x \in \mathbb{Z}$ define $T_x := \min\{n \ge 0 : S_n = x\}$ and suppose a, b are integers with a < 0 < b.
 - **2.1** Show that $T_a \wedge T_b := \min(T_a, T_b)$ is a stopping time and prove that $\mathsf{E}(T_a \wedge T_b)$ is finite.
 - **2.2** Prove that $(S_n)_{n\geq 0}$ is a martingale if and only if p = 1/2, and deduce that $\mathsf{P}(T_a < T_b) = b/(b-a)$ when p = 1/2.
 - **2.3** Now suppose p > 1/2 and let $Y_n = \beta^{S_n}$ for all $n \ge 0$. Find the value of $\beta \in (0,1)$ such that $(Y_n)_{n\ge 0}$ is a martingale with respect to $(S_n)_{n\ge 0}$.
 - **2.4** Hence calculate $\mathsf{P}(T_a < T_b)$ in terms of p > 1/2, a, and b.

You should show all your working and justify your calculations with suitable explanation.

Q3 Let $(X_n)_{n\geq 0}$, $X_0 = x$, be a random walk on the complete graph K_m on m > 2 vertices, such that at every step it jumps to any of the other m-1 vertices uniformly at random. Use an appropriate coupling to show that for every vertex $v \in K_m$ we have $|\mathsf{P}(X_n = v) - \frac{1}{m}| \leq e^{-an}$ for some $a = a_m > 0$. You should show all your working and justify your calculations with suitable explanation.



Q4 Let X(t), $t \ge 0$, be a continuous-time Markov chain on the state space $\{1, 2, 3\}$ whose generator (*Q*-matrix) is

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$$Q = \begin{pmatrix} -6 & 4 & 2\\ 0 & -2 & 2\\ 4 & 4 & -8 \end{pmatrix}.$$

- **4.1** Write down the backward Kolmogorov equations for the transition probabilities $p_{ij}(t), i, j \in \{1, 2, 3\}.$
- 4.2 Show that

$$P(t) = \exp\{tQ\} = \sum_{k \ge 0} \frac{t^k}{k!} Q^k$$

is a unique solution to the equations you obtained in question 4.1.

4.3 Define the resolvent $R(\lambda)$ for X(t) and find

$$p_{31}(t) = \mathsf{P}(X(t) = 1 \mid X(0) = 3).$$

You should show all your working and justify your calculations with suitable explanation.

- **Q5 5.1** Carefully define a Brownian motion, state its Markov and strong Markov properties.
 - 5.2 State the stopping theorem for bounded martingales.
 - Let $(B_t)_{t>0}$ be a Brownian motion starting at the origin $(B_0 = 0)$.
 - **5.3** Carefully show that for all $0 \le s < t$ we have

$$\mathsf{E}((B_t)^3 - 3tB_t | B_r, 0 \le r \le s) = (B_s)^3 - 3sB_s,$$
$$\mathsf{E}(e^{\theta B_t - t\theta^2/2} | B_r, 0 \le r \le s) = e^{\theta B_s - s\theta^2/2},$$

where θ is a real number, i.e., that $(B_t)^3 - 3tB_t$ and $e^{\theta B_t - t\theta^2/2}$ are martingales with respect to the natural filtration.

5.4 For a > 0, let $\tau = \inf\{t \ge 0 : |B_t| \ge a\}$ be the exit time from the interval (-a, a). Show that $\mathsf{E}(e^{-\beta\tau}) = 2/(e^{a\sqrt{2\beta}} + e^{-a\sqrt{2\beta}})$ for all $\beta \ge 0$.

You should show all your working and justify your calculations with suitable explanation.