Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2020	MATH3251-WE01

Title:

Stochastic Processes III

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. Show your working and explain your reasoning.

Revision:	

ED01/2020

Q1 **1.1** Carefully define the Poisson process $\big(N(t)\big)_{t\geq 0}$ with intensity $\lambda > 0$, and state its main properties.

1.2 For fixed $t > 0$, write the formula for the distribution of $N(t)$, i.e., for the values of probabilities $\mathsf{P}\big(N(t) = k\big)$ with integer $k \geq 0$. Prove your formula for the case $k = 0$.

1.3 For fixed $0 < t_1 < t_2$ and an integer $n \geq 0$, find the conditional probability

$$\mathsf{P}\big(N(t_1) = k \mid N(t_2) = n\big)\,, \qquad \text{where} \quad k = 0, 1, \dots, n\,.$$

Identify this distribution.

1.4 For integer $n > 0$, let T_n be the time of the nth event in $\big(N(t)\big)_{t\geq 0}$. Find the probability density function of T_{n+1} given $T_n = s$, and deduce that the conditional joint density of T_1, T_2 given $N_t = 2$ satisfies

$$f_{T_1,T_2|N_t}(s_1, s_2 \mid 2) = \begin{cases} \frac{2}{t^2}\,, & \text{if } 0 < s_1 < s_2 \leq t\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

1.5 Let X_1 and X_2 be independent with common uniform distribution, $X_i \sim \mathcal{U}(0, t]$. Define (Y_1, Y_2) as the order statistic for (X_1, X_2), namely, $Y_1 = \min(X_1, X_2)$ and $Y_2 = \max(Y_1, Y_2)$. Find the probability density function $f_{X_1,X_2}(x_1, x_2)$ and deduce that

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} \frac{2}{t^2}\,, & \text{if } 0 < y_1 < y_2 \leq t\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Deduce that $(T_1, T_2 \mid N_t = 2)$ from question **1.4** and (Y_1, Y_2) have the same distribution.

You should show all your working and justify your calculations with suitable explanation.

Q2 Let $X_1, X_2, \dots$ be independent identically distributed random variables with $\mathsf{P}(X_n = 1) = p = 1 - \mathsf{P}(X_n = -1)$ for some $p \in [1/2, 1)$. Set $S_0 = 0$ and $S_n = \sum_{i=1}^{n} X_i$. For $x \in \mathbb{Z}$ define $T_x := \min\{n \geq 0 : S_n = x\}$ and suppose a, b are integers with $a < 0 < b$.

2.1 Show that $T_a \wedge T_b := \min(T_a, T_b)$ is a stopping time and prove that $\mathsf{E}(T_a \wedge T_b)$ is finite.

2.2 Prove that $(S_n)_{n\geq 0}$ is a martingale if and only if $p = 1/2$, and deduce that $\mathsf{P}(T_a < T_b) = b/(b - a)$ when $p = 1/2$.

2.3 Now suppose $p > 1/2$ and let $Y_n = \beta^{S_n}$ for all $n \geq 0$. Find the value of $\beta \in (0, 1)$ such that $(Y_n)_{n\geq 0}$ is a martingale with respect to $(S_n)_{n\geq 0}$.

2.4 Hence calculate $\mathsf{P}(T_a < T_b)$ in terms of $p > 1/2$, a, and b.

You should show all your working and justify your calculations with suitable explanation.

Q3 Let $(X_n)_{n\geq 0}$, $X_0 = x$, be a random walk on the complete graph K_m on $m > 2$ vertices, such that at every step it jumps to any of the other $m-1$ vertices uniformly at random. Use an appropriate coupling to show that for every vertex $v \in K_m$ we have $|\mathsf{P}(X_n = v) - \frac{1}{m}| \leq \mathrm{e}^{-an}$ for some $a = a_m > 0$. You should show all your working and justify your calculations with suitable explanation.

Q4 Let $X(t)$, $t \geq 0$, be a continuous-time Markov chain on the state space $\{1, 2, 3\}$ whose generator (Q-matrix) is

$$Q = \begin{pmatrix} -6 & 4 & 2 \\ 0 & -2 & 2 \\ 4 & 4 & -8 \end{pmatrix}.$$

4.1 Write down the backward Kolmogorov equations for the transition probabilities $p_{ij}(t)$, i, $j \in \{1, 2, 3\}$.

4.2 Show that

$$P(t) = \exp\{tQ\} = \sum_{k \geq 0} \frac{t^k}{k!} Q^k$$

is a unique solution to the equations you obtained in question **4.1**.

4.3 Define the resolvent $R(\lambda)$ for $X(t)$ and find

$$p_{31}(t) = \mathsf{P}\big(X(t) = 1 \mid X(0) = 3\big).$$

You should show all your working and justify your calculations with suitable explanation.

Q5 **5.1** Carefully define a Brownian motion, state its Markov and strong Markov properties.

5.2 State the stopping theorem for bounded martingales.

Let $(B_t)_{t \geq 0}$ be a Brownian motion starting at the origin ($B_0 = 0$).

5.3 Carefully show that for all $0 \leq s < t$ we have

$$\mathsf{E}\big((B_t)^3 - 3tB_t | B_r, 0 \leq r \leq s\big) = (B_s)^3 - 3sB_s,$$
$$\mathsf{E}\big(e^{\theta B_t - t\theta^2/2} | B_r, 0 \leq r \leq s\big) = e^{\theta B_s - s\theta^2/2},$$

where θ is a real number, i.e., that $(B_t)^3 - 3tB_t$ and $e^{\theta B_t - t\theta^2/2}$ are martingales with respect to the natural filtration.

5.4 For $a > 0$, let $\tau = \inf\{t \geq 0 : |B_t| \geq a\}$ be the exit time from the interval $(-a, a)$. Show that $\mathsf{E}(e^{-\beta\tau}) = 2/\big(e^{a\sqrt{2\beta}} + e^{-a\sqrt{2\beta}}\big)$ for all $\beta \geq 0$.

You should show all your working and justify your calculations with suitable explanation.