Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2020	MATH3281-WE01

Title:
Topology III

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. Show your working and explain your reasoning.

Revision:	

ED01/2020

The following conventions hold in this paper

- $\mathbb{Z}$ denotes the additive group of all integer numbers with the discrete topology.
- $\mathbb{R}$ denotes the space of all real numbers with the standard topology.
- $\mathbb{C}$ denotes the space of all complex numbers with the standard topology.
- $\mathbb{R}^n$ denotes the real n-dimensional space with the standard topology.

Q1 **1.1** Define $d\colon \mathbb{R} \times \mathbb{R} \to [0, \infty)$ by

$$d(x, y) = \begin{cases} |x| + |y| & x \neq y \\ 0 & x = y \end{cases}.$$

Show that d is a metric on $\mathbb{R}$, and decide whether the topology induced by d agrees with the standard topology on $\mathbb{R}$.

1.2 Let $X = \{1, 2, 3\}$. Decide which of the following topologies on X are connected. Justify your statements.

(i) $\tau_1 = \{\emptyset, \{1, 2\}, \{1, 2, 3\}\}$.
(ii) $\tau_2 = \{\emptyset, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$.
(iii) $\tau_3 = \{\emptyset, \{1, 2\}, \{3\}, \{1, 2, 3\}\}$.

You can assume that these are indeed topologies.

1.3 Let A and B be finite one-dimensional simplicial complexes. Explain how to make $A \times B$ a two-dimensional simplicial complex. Use this to derive a formula relating the Euler characteristic of A, B and $A \times B$.

1.4 State, without proof, the Euler characteristic of S^1. Use your formula in (**1.3**) to compute the Euler characteristic of the torus.

1.5 Show that a finite tree is a contractible space. You may assume without proof that every finite tree with at least two vertices has a *free vertex*, i.e. a vertex that belongs to a single edge.

Q2 Let $\mathbb{N} = \{n \in \mathbb{Z} \mid n \geq 1\}$. For $n \in \mathbb{N}$ define

$$X_n = \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; \left(\frac{1}{n} - x\right)^2 + y^2 = \left(\frac{1}{n}\right)^2 \right\},$$
$$Y_n = \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; \left(\frac{1}{n} - x\right)^2 + y^2 \leq \left(\frac{1}{n}\right)^2 \right\},$$

and

$$Z_n = X_1 \cup \cdots \cup X_n \cup Y_{n+1}.$$

2.1 Show that each Z_n is connected and compact. Clearly state any results from the lectures that you use.

2.2 Let

$$X = \bigcup_{n=1}^{\infty} X_n \quad \text{and} \quad Z = \bigcap_{n=1}^{\infty} Z_n.$$

Show that $X = Z$, and that X is connected and compact. Again state any results from the lectures that you use.

2.3 Let

$$W = \{(x, y, n) \in \mathbb{R}^3 \mid (x, y) \in X_1, n \in \mathbb{N}\},$$

and let V be the quotient space $W/\sim$, where $\sim$ is the equivalence relation on W given by

(i) $(x, y, n) \sim (x, y, n)$ for all $(x, y, n) \in W$, and
(ii) $(0, 0, n) \sim (0, 0, m)$ for all $n, m \in \mathbb{N}$.

Show that V is connected and construct a bijective map $f \colon V \to X$. Justify why f is continuous.

2.4 Decide whether your f in (**2.3**) is a homeomorphism.

Q3 **3.1** State the definition of a topological group G, and the action of a topological group G on a topological space X.

3.2 Recall that

$$S^1 = \{z \in \mathbb{C} \mid z\bar{z} = 1\}$$

with complex multiplication is a topological group, where $\bar{z}$ denotes complex conjugation. Show that S^1 acts on

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$$

via

$$z \cdot (z_1, z_2) = (zz_1, \bar{z}z_2)$$

for $z \in S^1$ and $(z_1, z_2) \in S^3$.

3.3 Let $F\colon S^3 \to \mathbb{C} \times \mathbb{R}$ be given by

$$F(z_1, z_2) = (2z_1z_2, |z_1|^2 - |z_2|^2).$$

Show that the image

$$F(S^3) = S^2 = \{(z, t) \in \mathbb{C} \times \mathbb{R} \mid |z|^2 + t^2 = 1\},$$

and that F induces a well defined map $f\colon S^3/S^1 \to S^2$ which is a homeomorphism.

Q4 **4.1** Let X and Y be topological spaces. State the definition of a homotopy equivalence between X and Y. Let p be a point in X. State the definition of the fundamental group $\pi_1(X, p)$.

4.2 Let $X, Y \subset \mathbb{R}^3$ be two solid tori intersecting at exactly one point $p \in X \cap Y$ and consider their union $X \cup Y \subset \mathbb{R}^3$. Compute $\pi_1(X \cup Y, p)$. Justify your conclusions stating all the necessary results from the lectures without proof. You may use without proof the fundamental group of the circle.

4.3 Let X be a path connected topological space and let p_1, p_2 be two points in X. Show that $\pi_1(X, p_1)$ is isomorphic to $\pi_1(X, p_2)$.

Q5 **5.1** Let M, N be two compact surfaces without boundary. State the definition of the connected sum $M\#N$ of M and N.

5.2 Show that the Euler characteristic of the two-dimensional sphere is 2.

5.3 Let M be a connected compact surface without boundary. Show that, if M is orientable, then $\chi(M) = 2 - 2g$, where g is the number of torus summands in the connected sum decomposition of M given by the classification theorem for closed connected surfaces. Justify your conclusions stating all the necessary results from the lectures without proof.