

## EXAMINATION PAPER

Examination Session: May/June

Year: 2020

Exam Code:

MATH3291-WE01

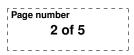
Title:

## Partial Differential Equations III

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks.
	Please start each question on a new page. Please write your CIS username at the top of each page.
	Show your working and explain your reasoning.

Revision:





**Q1** 1.1 Let (a, b) be a given interval, let  $f : [a, b] \to \mathbb{R}$  be a given continuous function and let  $t_1, t_2 \in \mathbb{R}$  be given. Consider the problem

$$\begin{cases} -u''(x) = f(x), & x \in (a, b), \\ u(a) = t_1; & u'(b) = t_2. \end{cases}$$

Find explicitly a Green's function  $G : [a, b] \times [a, b] \to \mathbb{R}$  such that the solution to the previous problem has the representation

$$u(x) = t_1 \partial_y G(x, a) + t_2 G(x, b) + \int_a^b G(x, y) f(y) \, \mathrm{d}y$$

**1.2** Let  $\Omega \subset \mathbb{R}^n$  be open, bounded and connected. We say that  $u \in C^2(\Omega) \cap C(\overline{\Omega})$  is subharmonic in  $\Omega$  if

$$-\Delta u \leq 0$$
 in  $\Omega$ .

Let  $\phi : \mathbb{R} \to \mathbb{R}$  be smooth and convex [i.e.  $f((1-t)x+ty) \leq (1-t)f(x)+tf(y)$  for all  $t \in [0,1]$  and for all  $x, y \in \mathbb{R}$ ]. Assume that u is harmonic and set  $v := \phi \circ u$ . Prove that v is subharmonic.

**1.3** Let  $f, g \in C^2(\mathbb{R})$ . Let  $u \in C^2(\mathbb{R} \times [0, \infty))$  be the solution

$$u(x,t) = \frac{1}{2} \left( f(x+t) + f(x-t) \right) + \frac{1}{2} \int_{x-t}^{x+t} g(y) \, dy$$

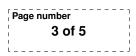
of the wave equation

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = 0, & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = f(x), \ u_t(x,0) = g(x), & x \in \mathbb{R}. \end{cases}$$

The kinetic energy k(t) and potential energy p(t) are given as

$$k(t) = \frac{1}{2} \int_{\mathbb{R}} u_t(x,t)^2 dx, \quad p(t) = \frac{1}{2} \int_{\mathbb{R}} u_x(x,t)^2 dx.$$

Suppose that f, g have compact support. Show that there exists  $t_0 > 0$  such that k(t) = p(t) for all  $t \ge t_0$ . *Hint:* Look at the difference k(t) - p(t).





Q2 Let us consider the problem

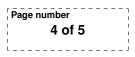
$$\begin{cases} \partial_t v + \frac{1}{2} (\partial_x v)^2 = 0, & \text{in } \mathbb{R} \times (0, +\infty), \\ v(x, 0) = v_0(x), & \text{in } \mathbb{R}, \end{cases}$$
(HJ)

where  $v_0 : \mathbb{R} \to \mathbb{R}$  is a given function which is twice continuously differentiable on  $\mathbb{R}$ . We aim to solve the problem by the method of characteristics.

- 2.1 Determine the type of the PDE appearing in (HJ) (i.e. is it linear; semi-linear; quasi-linear or fully nonlinear?). Justify your answer!
- **2.2** Identify the Cauchy data and the Cauchy curve, and give a parametrisation of it. Use the notation s for the parameter.
- **2.3** We will identify the system of ODEs satisfied by the flow  $\tau \mapsto (x(\tau, s), t(\tau, s))$ and the solution along the flow,  $\tau \mapsto z(\tau, s) := v(x(\tau, s), t(\tau, s))$ . In order to be able to solve this ODE system, one needs to introduce a new variable,  $\tau \mapsto p(\tau, s) := \partial_x v(x(\tau, s), t(\tau, s))$ . We rewrite the PDE in the form  $\partial_t v + \partial_x v \partial_x v = \frac{1}{2}(\partial_x v)^2$  and so one equation reads as  $\partial_\tau x(\tau, s) = p(\tau, s)$ . Find the ODEs satisfied by  $t(\tau, s), z(\tau, s)$  and  $p(\tau, s)$ . *Hint:* to find  $\partial_\tau p$ , differentiate the original PDE with respect to x.
- **2.4** Solve the new ODE system from **2.3** for (x, t, z, p).
- **2.5** Determine the maximal time  $t_{\text{max}} > 0$  for which the problem (HJ) has a classical solution on  $\mathbb{R} \times (0, t_{\text{max}})$ . *Hint:* depending on  $v''_0$ , for which values of t is the flow invertible?
- **2.6** Find a sufficient condition on  $v_0$  which allows to conclude that  $t_{\text{max}} = +\infty$  and therefore (HJ) has a global classical solution on  $\mathbb{R} \times (0, +\infty)$ .
- **2.7** For  $v_0(x) = \frac{1}{2}x^2$ , find the explicit solution to (HJ). What is the value of  $t_{\text{max}}$  in this case?
- Q3 We consider Burgers' equation

$$\begin{cases} \partial_t u + \frac{1}{2} \partial_x (u^2) = 0, & \text{in } \mathbb{R} \times (0, +\infty), \\ u(x, 0) = x, & \text{in } \mathbb{R}. \end{cases}$$
(Burgers')

- **3.1** Can one use a theorem presented during the lectures to conclude that (Burgers') has a global classical solution on  $\mathbb{R} \times (0, +\infty)$ ? Justify your answer!
- **3.2** Show that (Burgers') has a global classical solution and find explicitly this solution.





**Q4** 4.1 Let  $v \in L^1(\mathbb{R})$  and define  $\tau_a v \in L^1(\mathbb{R})$  by  $\tau_a v(x) = v(x-a)$ , which is the translation of v by  $a \in \mathbb{R}$ . Use a change of variables to prove that

$$\widehat{\tau_a v}(\xi) = e^{-i\xi a} \widehat{v}(\xi).$$

- **4.2** Let  $v \in C^1(\mathbb{R})$  such that  $v, v' \in L^1(\mathbb{R})$ . Show that  $\widehat{v'}(\xi) = i\xi \widehat{v}(\xi)$ .
- **4.3** Let  $g \in C(\mathbb{R})$ . Define  $G(x) := \int_0^x g(y) \, dy$ . Assume that  $g, G \in L^1(\mathbb{R})$ . Show that  $\widehat{G}(\xi) = \frac{1}{i\xi} \widehat{g}(\xi)$ .
- **4.4** Let  $f, g \in C^2(\mathbb{R})$ . Use the Fourier transform and parts **4.1**, **4.2**, **4.3** to derive the solution

$$u(x,t) = \frac{1}{2} \left( f(x+t) + f(x-t) \right) + \frac{1}{2} \int_{x-t}^{x+t} g(y) \, dy \tag{1}$$

of the wave equation

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = 0, & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = f(x), \ u_t(x,0) = g(x), & x \in \mathbb{R}. \end{cases}$$

No points will be given if the solution is found with a different method. *Hint:* First take the Fourier transform in x of u defined in (1).



**Q5** Let  $\Omega = (a_1, b_1) \times (a_2, b_2) \subset \mathbb{R}^2$  be bounded. Let  $u : \overline{\Omega} \times [0, \infty) \to \mathbb{R}$  be a smooth function satisfying

$$\begin{cases} u_t(\mathbf{x},t) - k\Delta u(\mathbf{x},t) = f(\mathbf{x}), & (\mathbf{x},t) \in \Omega \times (0,\infty), \\ u(\mathbf{x},t) = g(\mathbf{x}), & \mathbf{x} \in \partial\Omega \times [0,\infty), \\ u(\mathbf{x},0) = u_0(\mathbf{x}), & \mathbf{x} \in \Omega, \end{cases}$$

where  $u_0, f, g$  are smooth functions and k > 0. Let  $v : \overline{\Omega} \to \mathbb{R}$  be a smooth, time independent solution of the equation

$$\begin{cases} -k\Delta v(\mathbf{x}) = f(\mathbf{x}), & \mathbf{x} \in \Omega, \\ v(\mathbf{x}) = g(\mathbf{x}), & \mathbf{x} \in \partial \Omega. \end{cases}$$

Define the difference  $w(\mathbf{x}, t) = u(\mathbf{x}, t) - v(\mathbf{x})$ .

**5.1** Check that *w* satisfies

$$\begin{cases} w_t(\mathbf{x},t) - k\Delta w(\mathbf{x},t) = 0, & (\mathbf{x},t) \in \Omega \times (0,\infty), \\ w(\mathbf{x},t) = 0, & \mathbf{x} \in \partial\Omega \times [0,\infty), \\ w(\mathbf{x},0) = u_0(\mathbf{x}) - v(x), & \mathbf{x} \in \Omega. \end{cases}$$
(2)

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**5.2** Show that w satisfies the following Poincaré inequality on the rectangle  $\Omega$ : There exists  $C_{\Omega} > 0$  (independent of u, v) such that

$$\forall t \in (0,\infty): \quad \int_{\Omega} w(\mathbf{x},t)^2 \, d\mathbf{x} \le C_{\Omega} \int_{\Omega} |\nabla w(\mathbf{x},t)|^2 \, d\mathbf{x}. \tag{3}$$

*Hint:* You may use without proof the following one dimensional Poincaré inequality: There exists  $C_{a,b} > 0$  such that every function  $h \in C^1([a,b])$  with h(a) = h(b) = 0 satisfies  $\int_a^b h^2 dx \leq C_{a,b} \int_a^b (h')^2 dx$ .

**5.3** Define  $E(t) = \int_{\Omega} w(\mathbf{x}, t)^2 d\mathbf{x}$ . Use (2) and (3) to prove that there exists a constant C > 0 (independent of u, v) such that

$$\forall t \in (0, \infty) : \quad E'(t) \le -CE(t).$$

5.4 State Grönwall's inequality (without proof) and use it to prove that

$$\lim_{t \to \infty} \int_{\Omega} (u(\mathbf{x}, t) - v(\mathbf{x}))^2 \, d\mathbf{x} = 0,$$

which means that the solution u converges to the time independent solution v (with respect to the  $L^2(\Omega)$  norm).