Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2020	MATH3291-WE01

Title:
Partial Differential Equations III

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. Show your working and explain your reasoning.

Revision:	

ED01/2020

Q1 **1.1** Let (a,b) be a given interval, let $f:[a,b]\to\mathbb{R}$ be a given continuous function and let $t_1,t_2\in\mathbb{R}$ be given. Consider the problem

$$\begin{cases} -u''(x)=f(x), & x\in(a,b),\\ u(a)=t_1;\ u'(b)=t_2. \end{cases}$$

Find explicitly a Green's function $G:[a,b]\times[a,b]\to\mathbb{R}$ such that the solution to the previous problem has the representation

$$u(x)=t_1\partial_y G(x,a)+t_2 G(x,b)+\int_a^b G(x,y)f(y)\,\mathrm{d}y.$$

1.2 Let $\Omega\subset\mathbb{R}^n$ be open, bounded and connected. We say that $u\in C^2(\Omega)\cap C(\overline{\Omega})$ is subharmonic in Ω if

$$-\Delta u\le 0 \quad \text{in} \quad \Omega.$$

Let $\phi:\mathbb{R}\to\mathbb{R}$ be smooth and convex [i.e. $f((1-t)x+ty)\le(1-t)f(x)+tf(y)$ for all $t\in[0,1]$ and for all $x,y\in\mathbb{R}$]. Assume that u is harmonic and set $v:=\phi\circ u$. Prove that v is subharmonic.

1.3 Let $f,g\in C^2(\mathbb{R})$. Let $u\in C^2(\mathbb{R}\times[0,\infty))$ be the solution

$$u(x,t)=\frac{1}{2}\big(f(x+t)+f(x-t)\big)+\frac{1}{2}\int_{x-t}^{x+t}g(y)\,dy$$

of the wave equation

$$\begin{cases} u_{tt}(x,t)-u_{xx}(x,t)=0, & (x,t)\in\mathbb{R}\times(0,\infty),\\ u(x,0)=f(x),\ u_t(x,0)=g(x), & x\in\mathbb{R}. \end{cases}$$

The kinetic energy $k(t)$ and potential energy $p(t)$ are given as

$$k(t)=\frac{1}{2}\int_{\mathbb{R}}u_t(x,t)^2\,dx,\quad p(t)=\frac{1}{2}\int_{\mathbb{R}}u_x(x,t)^2\,dx.$$

Suppose that f,g have compact support. Show that there exists $t_0>0$ such that $k(t)=p(t)$ for all $t\ge t_0$. *Hint:* Look at the difference $k(t)-p(t)$.

Q2 Let us consider the problem

$$\begin{cases} \partial_t v + \frac{1}{2}(\partial_x v)^2 = 0, & \text{in } \mathbb{R} \times (0, +\infty), \\ v(x, 0) = v_0(x), & \text{in } \mathbb{R}, \end{cases} \tag{HJ}$$

where $v_0 : \mathbb{R} \to \mathbb{R}$ is a given function which is twice continuously differentiable on $\mathbb{R}$. We aim to solve the problem by the method of characteristics.

2.1 Determine the type of the PDE appearing in (HJ) (i.e. is it linear; semi-linear; quasi-linear or fully nonlinear?). Justify your answer!

2.2 Identify the Cauchy data and the Cauchy curve, and give a parametrisation of it. Use the notation s for the parameter.

2.3 We will identify the system of ODEs satisfied by the flow $\tau \mapsto (x(\tau, s), t(\tau, s))$ and the solution along the flow, $\tau \mapsto z(\tau, s) := v(x(\tau, s), t(\tau, s))$. In order to be able to solve this ODE system, one needs to introduce a new variable, $\tau \mapsto p(\tau, s) := \partial_x v(x(\tau, s), t(\tau, s))$. We rewrite the PDE in the form $\partial_t v + \partial_x v \partial_x v = \frac{1}{2}(\partial_x v)^2$ and so one equation reads as $\partial_\tau x(\tau, s) = p(\tau, s)$. Find the ODEs satisfied by $t(\tau, s)$, $z(\tau, s)$ and $p(\tau, s)$. *Hint:* to find $\partial_\tau p$, differentiate the original PDE with respect to x.

2.4 Solve the new ODE system from **2.3** for (x, t, z, p).

2.5 Determine the maximal time $t_{\max} > 0$ for which the problem (HJ) has a classical solution on $\mathbb{R} \times (0, t_{\max})$. *Hint:* depending on v_0'', for which values of t is the flow invertible?

2.6 Find a sufficient condition on v_0 which allows to conclude that $t_{\max} = +\infty$ and therefore (HJ) has a global classical solution on $\mathbb{R} \times (0, +\infty)$.

2.7 For $v_0(x) = \frac{1}{2}x^2$, find the explicit solution to (HJ). What is the value of $t_{\max}$ in this case?

Q3 We consider Burgers' equation

$$\begin{cases} \partial_t u + \frac{1}{2}\partial_x(u^2) = 0, & \text{in } \mathbb{R} \times (0, +\infty), \\ u(x, 0) = x, & \text{in } \mathbb{R}. \end{cases} \tag{Burgers'}$$

3.1 Can one use a theorem presented during the lectures to conclude that (Burgers') has a global classical solution on $\mathbb{R} \times (0, +\infty)$? Justify your answer!

3.2 Show that (Burgers') has a global classical solution and find explicitly this solution.

Q4 **4.1** Let $v \in L^1(\mathbb{R})$ and define $\tau_a v \in L^1(\mathbb{R})$ by $\tau_a v(x) = v(x-a)$, which is the translation of v by $a \in \mathbb{R}$. Use a change of variables to prove that

$$\widehat{\tau_a v}(\xi) = e^{-\mathrm{i}\xi a}\widehat{v}(\xi).$$

4.2 Let $v \in C^1(\mathbb{R})$ such that $v, v' \in L^1(\mathbb{R})$. Show that $\widehat{v'}(\xi) = \mathrm{i}\xi\widehat{v}(\xi)$.

4.3 Let $g \in C(\mathbb{R})$. Define $G(x) := \int_0^x g(y)\,dy$. Assume that $g, G \in L^1(\mathbb{R})$. Show that $\widehat{G}(\xi) = \frac{1}{\mathrm{i}\xi}\widehat{g}(\xi)$.

4.4 Let $f, g \in C^2(\mathbb{R})$. Use the Fourier transform and parts **4.1**, **4.2**, **4.3** to derive the solution

$$u(x,t) = \frac{1}{2}\big(f(x+t) + f(x-t)\big) + \frac{1}{2}\int_{x-t}^{x+t} g(y)\,dy \tag{1}$$

of the wave equation

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = 0, & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = f(x),\ u_t(x,0) = g(x), & x \in \mathbb{R}. \end{cases}$$

No points will be given if the solution is found with a different method. *Hint:* First take the Fourier transform in x of u defined in (1).

Q5 Let $\Omega = (a_1, b_1) \times (a_2, b_2) \subset \mathbb{R}^2$ be bounded. Let $u : \overline{\Omega} \times [0, \infty) \to \mathbb{R}$ be a smooth function satisfying

$$\begin{cases} u_t(\mathbf{x}, t) - k\Delta u(\mathbf{x}, t) = f(\mathbf{x}), & (\mathbf{x}, t) \in \Omega \times (0, \infty), \\ u(\mathbf{x}, t) = g(\mathbf{x}), & \mathbf{x} \in \partial\Omega \times [0, \infty), \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}), & \mathbf{x} \in \Omega, \end{cases}$$

where u_0, f, g are smooth functions and $k > 0$. Let $v : \overline{\Omega} \to \mathbb{R}$ be a smooth, time independent solution of the equation

$$\begin{cases} -k\Delta v(\mathbf{x}) = f(\mathbf{x}), & \mathbf{x} \in \Omega, \\ v(\mathbf{x}) = g(\mathbf{x}), & \mathbf{x} \in \partial\Omega. \end{cases}$$

Define the difference $w(\mathbf{x}, t) = u(\mathbf{x}, t) - v(\mathbf{x})$.

5.1 Check that w satisfies

$$\begin{cases} w_t(\mathbf{x}, t) - k\Delta w(\mathbf{x}, t) = 0, & (\mathbf{x}, t) \in \Omega \times (0, \infty), \\ w(\mathbf{x}, t) = 0, & \mathbf{x} \in \partial\Omega \times [0, \infty), \\ w(\mathbf{x}, 0) = u_0(\mathbf{x}) - v(x), & \mathbf{x} \in \Omega. \end{cases} \tag{2}$$

5.2 Show that w satisfies the following Poincaré inequality on the rectangle Ω: There exists $C_\Omega > 0$ (independent of u, v) such that

$$\forall\, t \in (0, \infty) : \quad \int_\Omega w(\mathbf{x}, t)^2 \, d\mathbf{x} \le C_\Omega \int_\Omega |\nabla w(\mathbf{x}, t)|^2 \, d\mathbf{x}. \tag{3}$$

Hint: You may use without proof the following one dimensional Poincaré inequality: There exists $C_{a,b} > 0$ such that every function $h \in C^1([a, b])$ with $h(a) = h(b) = 0$ satisfies $\int_a^b h^2 \, dx \le C_{a,b} \int_a^b (h')^2 \, dx$.

5.3 Define $E(t) = \int_\Omega w(\mathbf{x}, t)^2 \, d\mathbf{x}$. Use (2) and (3) to prove that there exists a constant $C > 0$ (independent of u, v) such that

$$\forall\, t \in (0, \infty) : \quad E'(t) \le -CE(t).$$

5.4 State Grönwall's inequality (without proof) and use it to prove that

$$\lim_{t \to \infty} \int_\Omega (u(\mathbf{x}, t) - v(\mathbf{x}))^2 \, d\mathbf{x} = 0,$$

which means that the solution u converges to the time independent solution v (with respect to the $L^2(\Omega)$ norm).