EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2020	MATH3301-WE01

Title:
Mathematical Finance III

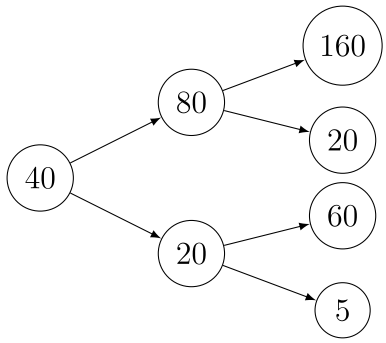
Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. Show your working and explain your reasoning.

Revision:	

ED01/2020

Q1 **1.1** Consider the following 2-period Binomial market shown as a tree.



The interest rate is 1/2. Answer the following questions and show your work.

(i) Calculate the martingale probabilities along the edges of the tree.

(ii) Consider the contingent claim $X = \min\{S_0, S_1, S_2\}$. Calculate the arbitrage-free price of X at every node of the tree.

(iii) Explain in words the meaning of the arbitrage-free price of X at the node with 80.

1.2 (i) State the Itô isometry for a general Itô integral.

In the next two parts of this question, $(W_t)_{t \geq 0}$ is a Brownian motion.

(ii) Using question **1.2**(i) and the moment generating function of the normal distribution, or otherwise, compute $\mathbb{E}\Big[\big(\int_0^1 \sqrt{\cosh W_t}\, \mathrm{d}W_t\big)^2\Big]$.

(iii) Let $I = \int_0^1 W_t\, \mathrm{d}t$. Show that $\mathbb{E}[I^2] = \int_0^1 \int_0^1 \mathbb{E}[W_s W_t] \mathrm{d}s \mathrm{d}t$.
Hence find $\mathbb{V}\mathrm{ar}(I)$ and $\mathbb{C}\mathrm{ov}(I, W_1)$.

Q2 Consider the following contingent claim. There is a stock whose value at time t is S_t. You have the option, but not the obligation, to buy 2 units of this stock for K units of money at time t.

2.1 Consider a T-period Binomial market (B_t, S_t) with interest rate r and no arbitrage. Let X be the value of the contingent claim above at expiry time T. Show, with appropriate explanation, that $X = (2S_T - K)_+$.

2.2 Consider the American option for exercising the contingent claim above, which means you can exercise it at any stopping time τ with $0 \leq \tau \leq T$. Prove that the arbitrage-free price of this contingent claim with the American exercising option is the same as its arbitrage-free price with the European exercising option.

Q3 Consider a 1-period market $\mathcal{M} = (B_t, S_t^1, S_t^2)$ such that

- $B_0 = 1$ and $B_1 = 1.1$;
- $(S_0^1, S_0^2) = (10, 20)$ and (S_1^1, S_1^2) has the following joint distribution:

$$(S_1^1, S_1^2) = \begin{cases} (15, 21) & \text{with probability } 0.5, \\ (10, 22.5) & \text{with probability } 0.25, \\ (10, 22.2) & \text{with probability } 0.25. \end{cases}$$

A portfolio for this market is a vector $h = (x, y, z) \in \mathbb{R}^3$ and its value is $V_t^h = xB_t + yS_t^1 + zS_t^2$ for $t = 0, 1$. The market contains arbitrage if there is a portfolio h such that (i) $V_0^h = 0$, (ii) $V_1^h \geq 0$ almost surely and (iii) $V_1^h > 0$ with positive probability.

3.1 Prove this market contains no arbitrage.

3.2 Consider a contingent claim $X = F(S_1^1, S_1^2)$. Show that there is a portfolio h^* such that $V_1^{h^*} = X$ almost surely. (You need not find h^* explicitly but you must justify why such an h^* exists.)

3.3 Prove that the arbitrage-free price of X at time 0 is $V_0^{h^*}$.

Q4 **4.1** State Itô's Lemma for $f(t, X_t)$ where f is smooth and $(X_t)_{t\geq 0}$ is an Itô process.

Consider an Itô process $(X_t)_{t\geq 0}$ with $X_t \geq 0$ satisfying the following stochastic differential equation (SDE) involving the constant parameter $c > 0$:

$$X_0 = 0; \quad \mathrm{d}X_t = c\,\mathrm{d}t + \sqrt{X_t}\,\mathrm{d}W_t, \ t \geq 0.$$

4.2 Let k be a positive integer. Apply Itô's Lemma to find an SDE of the form

$$\mathrm{d}(X_t^k) = aX_t^{k-1}\mathrm{d}t + bX_t^{k-(1/2)}\mathrm{d}W_t, \ t \geq 0,$$

for constants a, b, depending on c and k, that you should determine.

4.3 Use your SDE from question **4.2** to explain why $\mathbb{E}(X_t^k) = a\int_0^t \mathbb{E}(X_s^{k-1})\mathrm{d}s$.

4.4 Use the formula from question **4.3** and mathematical induction to show that

$$\mathbb{E}(X_t^k) = t^k \prod_{j=0}^{k-1}\left(c + \frac{j}{2}\right).$$

4.5 If $(W_t)_{t\geq 0}$ is Brownian motion, write down an SDE for $Y_t = \frac{1}{4}W_t^2$.
Use the earlier parts of this question to deduce $\mathbb{E}(W_t^{2k})$ for positive integer k.
Hence, without doing any further calculations, if $Z \sim \mathcal{N}(0, 1)$, what is $\mathbb{E}(Z^4)$?

Q5 Consider a Black–Scholes market with risk-free asset given by $B_t = \mathrm{e}^{rt}$, $r > 0$, and two risky assets with evolution given by $S_0^{(1)} = S_0^{(2)} = 1$ and

$$\mathrm{d}S_t^{(1)} = \mu_1 S_t^{(1)}\mathrm{d}t + \sigma_1 S_t^{(1)}\mathrm{d}W_t^{(1)}, \quad \text{and} \quad \mathrm{d}S_t^{(2)} = \mu_2 S_t^{(2)}\mathrm{d}t + \sigma_2 S_t^{(2)}\mathrm{d}W_t^{(2)}.$$

Here $\mu_1, \mu_2, \sigma_1, \sigma_2$ are positive constants, and

$$W_t^{(1)} = W_t, \quad \text{and} \quad W_t^{(2)} = \rho W_t + \sqrt{1-\rho^2}W_t',$$

where $\rho \in (-1, +1)$ is constant and W_t, W_t' are independent Brownian motions under the real-world measure $\mathbb{P}$.
For $\gamma \in \mathbb{R}$, let $R_t = S_t^{(1)} \left(S_t^{(2)}\right)^\gamma$.

5.1 Using Itô's formula, derive SDEs for $L_t^{(1)} = \log S_t^{(1)}$ and $L_t^{(2)} = \log S_t^{(2)}$.

5.2 Using question **5.1**, or otherwise, write down an SDE for $\log R_t$. Deduce that

$$\mathrm{d}R_t = \mu R_t \mathrm{d}t + \sigma R_t \mathrm{d}W_t'',$$

where $W_t'' = \alpha W_t + \sqrt{1-\alpha^2}W_t'$, and μ, σ, and α are constant functions of μ_1, μ_2, σ_1, σ_2, ρ, and γ that you should determine.

5.3 If $\mathbb{Q}$ is a measure under which $\mathrm{e}^{-rt}R_t$ is a martingale, find an expression for $\mathbb{Q}(R_T \geq \theta \mid \mathcal{F}_t)$, $0 \leq t \leq T$, in terms of the standard normal cumulative distribution function.

Consider the contingent claim $X = \Phi(S_1^{(1)}, S_1^{(2)}) = \mathbf{1}\{S_1^{(1)} \geq 2S_1^{(2)}\}$.

5.4 Use your answer to question **5.3** to find $\Pi_t(X) = C(t, S_t^{(1)}, S_t^{(2)})$, the no-arbitrage price of X at time $t \in [0, 1]$, in terms of a function $C(t, x, y)$ that you should determine. In particular, using also, where necessary, your formulas for μ and σ derived in question **5.2**, compute $\Pi_0(X)$ in the case where $\mu_1 = 1.2$, $\mu_2 = 1.9$, $\sigma_1 = 1.8$, $\sigma_2 = 0.2$, $r = 0.05$, and $\rho = -0.7$.

In your answer to question **5.4** you may use the following table of values of the standard normal cumulative distribution function.

z	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$N(z)$	0.540	0.579	0.618	0.655	0.691	0.726	0.758	0.788	0.816	0.841
z	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
$N(z)$	0.864	0.885	0.903	0.919	0.933	0.945	0.955	0.964	0.971	0.977
z	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3.0
$N(z)$	0.982	0.986	0.989	0.992	0.994	0.995	0.997	0.997	0.998	0.999