EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2020	MATH3341-WE01

Title:
Bayesian Statistics III

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. Show your working and explain your reasoning.

Revision:	

ED01/2020

Q1 **1.1** Let θ denote the average recovery time (in days) for patients with a disease. You are interested in specifying a suitable prior distribution for θ, and you wish to elicit it based on expert knowledge. Your expert judges that $\phi = \theta^\alpha$, for some $\alpha > 0$, has an exponential distribution with cumulative distribution function

$$F(\phi) = \begin{cases} 1 - \exp(-\beta\phi) & \text{for } \phi \in (0, +\infty) \\ 0 & \text{otherwise} \end{cases}$$

for some $\beta > 0$. In addition, from his graduate school studies, he judges that there is 25% probability that $\theta < 8$ days and 25% probability that $\theta > 15$ days. Specify the prior distribution for θ that you can use in your analysis by stating its probability density function and computing its hyper-parameter values.

1.2 Suppose that $y_1, ...y_n$ are a random sample from a uniform distribution on the range $(0, \theta)$. Assume that $\gamma = \log(\theta)$ admits a Normal prior distribution with known mean μ and variance σ^2. Calculate the posterior distribution of γ given the observations, including an explicit formula for the normalising constant.

Hint: $n\gamma + \frac{1}{2\sigma^2}(\gamma - \mu)^2 = \frac{1}{2\sigma^2}(\gamma - \mu + n\sigma^2)^2 + n\mu - \frac{1}{2}n^2\sigma^2$

1.3 An important theorem in relation to Bayesian networks states that: *If A, B and S are collections of random quantities in a Bayesian network, then A and B are conditionally independent given S if A and B are separated by S in the moral graph of an ancestral subgraph containing $A \cup B \cup S$ of the directed acyclic graph of the Bayesian network.*

(i) Consider a joint distribution for 7 random quantities which can be written in the form

$$p(a, b, c, d, e, f, g) = p(a)p(b)p(c \mid a, b)p(d \mid b)p(e \mid c, d)p(f \mid e, g)p(g \mid e).$$

Draw the DAG representing this factorization.

(ii) Illustrate the application of the theorem by enumerating all the sets of quantities V for which the theorem implies that $c \perp d \mid V$.

1.4 Use Laplace's method to derive an approximation for the normalizing constant of the following non-normalized posterior distribution of parameter $\theta \in \mathbb{R}$

$$f(\theta|y) \propto e^{(\alpha+y)\theta}e^{-\beta e^\theta},$$

where $y \in \mathbb{R}$, $\alpha > 0$ and $\beta > 0$.

Q2 Let $y = (y_1, ..., y_n)$ be a sequence of n observables assumed to be iid according to a Log-Normal sampling distribution with parameters $\mu \in \mathbb{R}$ and $\sigma^2 \in (0, +\infty)$; i.e.

$$y_i|\mu, \sigma^2 \overset{\text{iid}}{\sim} \text{LN}(\mu, \sigma^2) \qquad , i = 1, ..., n$$

where μ is an unknown parameter, and σ^2 is assumed known.

Hint: The Log-Normal distribution denoted by $\text{LN}(\mu, \sigma^2)$ has density function

$$f(x|\mu, \sigma^2) = \begin{cases} \frac{1}{\sqrt{2\pi\sigma^2}} \frac{1}{x} \exp\left(-\frac{1}{2}\frac{(\log(x)-\mu)^2}{\sigma^2}\right) & , \text{if } x \in (0, +\infty) \\ 0 & , \text{otherwise} \end{cases}$$

2.1 Show that the LN distribution with known σ^2 is an exponential family of distributions.

2.2 Compute the likelihood of y given (μ, σ^2) and its sufficient statistic.

2.3 Derive the prior distribution which is conjugate to the likelihood function of the problem.

2.4 Using the conjugate prior, construct the $(1 - a)100\%$ HPD posterior credible interval for μ, and show your working. Compute the bounds of the 95% HPD posterior credible interval for μ, when there is available a data-set $y = (0.05, 0.36, 0.13, 0.22, 0.60)$ of size $n = 5$ observations; $\sigma^2 = 1$; the prior mean of μ is 0; and the prior variance of μ is 10.

Hint: The 0.975-quantile of the standard Normal distribution is $z^*_{0.975} = 1.959964$.

Q3 **3.1** Consider random variables $\theta \sim \mathrm{N}(0,1)$ and $y \mid \theta \sim \mathrm{N}(\theta, 1)$, and a loss function $\ell(\theta, d) = (\theta - d)^2$ for all $d \in \mathbb{R}$ and $\theta \in \mathbb{R}$. We denote as $\mathrm{N}(\mu, \sigma^2)$, the Normal distribution with mean μ and variance σ^2.

(i) Consider a decision rule δ_a, where $\delta_a(y) = ay$ for all $y \in \mathbb{R}$, and where $a \in \mathbb{R}$ is an arbitrary constant. Show that the (Frequentist) risk function for decision δ_a is

$$R(\theta, \delta_a) = (1-a)^2\theta^2 + a^2.$$

(ii) Show that δ_a is inadmissible when $a < 0$ or $a > 1$.

(iii) Compute the Bayesian point estimator of θ under the aforesaid loss function and Bayesian model. State if this estimator is admissible and justify your answer.

3.2 Assume a 1-dimensional random quantity $x \sim Q(x|y)$, with unimodal density $q(x|y)$. Show that the $(1-a)$-credible interval $C_a = [L, U]$ for x as a Bayesian rule C_a under the loss function

$$\ell(x, C_a; L, U) = k(U - L) - 1(x \in [L, U]), \quad \text{with} \quad k \in (0, \max_{\forall x \in \mathbb{R}}(q(x|y)))$$

is given by $q(L) = q(U) = k$, and $\mathsf{P}_Q(x \in [L, U]|y) = 1 - a$. Discuss known properties of the derived credible interval.

Q4 **4.1** Consider the following probability density function, which is known up to a constant of proportionality

$$f(x) = \frac{e^x}{c},$$

where $x \in [0, 1]$. Assuming the numbers below are a sequence of independent random numbers uniformly distributed on $[0, 1]$, generate 3 values from $f(x)$ using inverse sampling.

0.156 0.579 0.936

4.2 The Kumaraswamy distribution is a flexible alternative to the beta distribution. The probability density function of this distribution is given by

$$f(x) = \alpha\beta x^{(\alpha-1)}(1 - x^{\alpha})^{(\beta-1)},$$

where $x \in (0, 1)$, $\alpha > 0$ and $\beta > 0$. Assuming that $\alpha = \beta = 2$, perform 3 iterations of rejection sampling from $f(x)$ using a uniform distribution on $[0, 1]$ as proposal distribution. State whether the generated values are accepted or not. Base your calculations on the following sequence of uniformly distributed random numbers between 0 and 1:

0.046 0.495 0.307 0.138 0.645 0.515

4.3 We want to use Gibbs sampling to sample from the joint distribution of A and B, which has probabilities proportional to the table below.

		B = 1	B = 2	B = 3
A	4	0.4	0.5	0.8
	5	0.4	0.6	0.5
	6	0.5	0.7	0.9

Generate the output of the Gibbs sampler assuming availability of the sequence of uniform random numbers in $[0, 1]$ given below and using as initial value $B^{(0)} = 2$.

0.963 0.801 0.526 0.039 0.101 0.675

Q5 A colony of insects is studied with a view to quantifying variation in the number of eggs laid and in the rate at which eggs successfully develop. Let N_i denote the number of eggs laid by insect i, for $i = 1, \ldots, n$, and E_i the number of eggs which develop. Eggs laid by insect i develop independently with probability of success p_i. Expert judgment is that the variability between insects in numbers of eggs laid should be modelled well by a Poisson distribution with fixed rate λ. Furthermore, it is thought that a beta distribution with fixed parameters α and β would adequately describe variability between insects with respect to p_i.

5.1 Draw a directed acyclic graph using plate notation for the Bayesian network describing the joint distribution of the $\{N_i\}$, $\{E_i\}$ and $\{p_i\}$ based on the above, and add vertices for the parameters λ, α and β.

5.2 Specify the distributions for the vertices $\{N_i\}$, $\{E_i\}$ and $\{p_i\}$ given their respective parents.

5.3 On closer inspection, the experts are uneasy about the choice of fixed values of the parameters. Therefore, the model is modified by assigning exponential prior distributions with rate 1 for the parameters α and β of the beta distribution and an improper prior proportional to $1/\lambda$ for λ. Derive (up to multiplicative constants) all the conditional distributions required for Gibbs sampling. Which of the conditional distributions are known distributions and what are their parameters?

5.4 Describe an efficient approach to generating values from each of the conditional distributions in part **5.3**. You may refer to standard functions in `R` or name standard algorithms. For any algorithm you name, show that preconditions (if any) for its application are met. You may use the fact that the third derivative of $\log \Gamma(x)$ is negative for all positive x, where $\Gamma(\cdot)$ is the gamma function.

Hint-1: The Poisson distribution for $x \in \{0, 1, \ldots\}$ with parameter λ takes the form

$$P(x|\lambda) = e^{-\lambda} \frac{\lambda^x}{x!}.$$

Hint-2: The pdf of a beta-distributed random quantity $x \in (0, 1)$ with parameters a and b is

$$f(x|a, b) = \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1 - x)^{b-1}.$$

Hint-3: The exponential distribution for $x \in [0, \infty)$ with parameter θ takes the form

$$f(x|\theta) = \theta e^{-\theta x}.$$