Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2020	MATH3391-WE01

Title:
Quantum Information III

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. Show your working and explain your reasoning.

Revision:	

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Q1 **1.1** The density matrix ρ for a single qubit can be described in terms of the Bloch sphere as

$$\rho = \frac{1}{2}\left(I + \mathbf{r} \cdot \sigma\right)$$

where the Bloch vector $\mathbf{r}$ is a position vector in three dimensions, I is the identity matrix and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(i) Given two density matrices ρ_1 and ρ_2 with corresponding Bloch vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, compute the trace $\mathrm{Tr}\left(\rho_1\rho_2\right)$ in terms of the two Bloch vectors.

(ii) What does it mean geometrically (in terms of vectors in the Bloch sphere) that the commutator of the two density matrices ρ_1 and ρ_2 vanishes, i.e. $[\rho_1, \rho_2] = \rho_1\rho_2 - \rho_2\rho_1 = 0$?

1.2 The operator

$$M = \frac{1}{2}\left(I + X_0X_1 + Y_0Y_1 + Z_0Z_1\right)$$

acts on 2 qubits, where X_i, Y_i and Z_i represent the Pauli operators acting on qubit i.

(i) Calculate the result of M acting on each of the computational basis states $|q_1q_0\rangle$.

(ii) What is the action of M on an arbitrary separable 2-qubit state $|\psi\rangle\,|\phi\rangle$?

(iii) Show that $M^2 = I$.

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Q2 Consider a bipartite system where Alice has two qubits which we will label as system AB, and Charlie has one qubit which we label as system C.

2.1 If the system is in a separable pure state, write an expression for the most general form of the state and hence give an expression for the density operator $\hat{\rho}$. Calculate the reduced density operator of system AB, $\hat{\rho}_{AB} \equiv \mathrm{Tr}_C(\hat{\rho})$ and evaluate $\mathrm{Tr}(\hat{\rho}_{AB})$ and $\mathrm{Tr}(\hat{\rho}^2_{AB})$.

2.2 For each of the following states

$$
\begin{aligned}
|\Psi\rangle &= \frac{1}{\sqrt{3}}\left(|100\rangle + |010\rangle + |001\rangle\right) \\
|\Phi\rangle &= \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)
\end{aligned}
$$

calculate the reduced density operator in system AB, and give an interpretation as an ensemble of orthonormal pure states.

Give an example of an observable $\hat{M}$ Charlie could measure which would result in these states in system AB with the same probabilities as in the ensembles you found.

2.3 Suppose Charlie measures the observable $\hat{N} = |0\rangle\langle 1| + |1\rangle\langle 0|$. For each of the initial states $|\Psi\rangle$ and $|\Phi\rangle$, what are the possible final states in system AB, and the probabilities for each outcome?

2.4 If Alice sends her second qubit to Bob, we can consider the system AB as a bipartite system where Alice has one qubit and Bob the other.

For each of the states $|\Psi\rangle$ and $|\Phi\rangle$, explain whether or not it is possible for Charlie to make a local measurement so that with probability 1 the resulting state in system AB:

- is entangled.
- is separable.

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Q3 Consider a single qubit Hilbert space with orthonormal basis states $|0\rangle$ and $|1\rangle$ and define

$$|\Phi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\phi}|1\rangle\right) ,$$

with $\phi \in [0, 2\pi]$.

3.1 Compute the corresponding density matrix ρ using the Bloch representation

$$\rho = \frac{1}{2}\left(I + \mathbf{r}\cdot\sigma\right) .$$

Describe geometrically, i.e. using the Bloch sphere, the location of the corresponding Bloch vector.

3.2 Compute $U(\alpha) = e^{-i\alpha\sigma_3}$ and show that it is a unitary operator for $\alpha \in \mathbb{R}$.

3.3 Find the Bloch vector for the density matrix $\rho' = U(\alpha)\rho U(\alpha)^\dagger$ for ρ computed in part (a). Describe geometrically, i.e. using the Bloch sphere, your results.

3.4 Consider now two particular states

$$|\pm\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \pm |1\rangle\right) ,$$

corresponding to $\phi = 0$ and $\phi = \pi$ respectively, and denote with $\rho_\pm$ the associated density matrices in Bloch representation. Use the geometric interpretation derived in point (c) to deduce the Bloch vectors for the evolved density matrices $\rho'_\pm = U(\frac{\pi}{4})\rho_\pm U(\frac{\pi}{4})^\dagger$.

3.5 Suppose Alice and Bob each have one qubit of a bipartite system in state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle\otimes|-\rangle + |-\rangle\otimes|+\rangle\right) .$$

Alice and Bob decide to evolve their systems with local unitary operators. Alice implements $U(\frac{\pi}{4})$ while Bob uses σ_1. Does it matter who evolves first? What is the state $|\chi\rangle$ of the system after these evolutions? If now Alice measures $Y = \sigma_2$ on the evolved state $|\chi\rangle$ what are the possible values, the corresponding probabilities, and corresponding final states?

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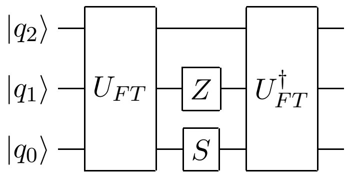
Q4 Consider the Quantum Fourier Transform, defined as the linear operator U_{FT} acting on an n-qubit Hilbert space, whose action on orthonormal (computational) basis states $|x\rangle$, $x \in \{0, 1, \ldots, 2^n - 1\}$ is

$$U_{FT}|x\rangle = \frac{1}{2^{n/2}} \sum_{y=0}^{2^n-1} e^{2\pi i xy/N} |y\rangle \ ,$$

where $N = 2^n$.

4.1 Using the definition above, show that the states $U_{FT}|x\rangle$ are orthonormal. What does this say about the operator U_{FT}?

4.2 Consider a 3-qubit system, and consider the unitary transform $U_{FT}^{\dagger} S_0 Z_1 U_{FT}$, represented by the quantum circuit below.



Show that this circuit implements the operation $x \to x + 2 \mod 8$ for input basis states $|x\rangle$.

Hint: Write the computational basis states in the output of the Quantum Fourier Transform in terms of their bits, i.e. $y = 4y_2 + 2y_1 + y_0$ to find the action on each qubit.

4.3 Suppose we want to implement the operation $x \to x + w \mod 8$ for some integer w using a unitary transformation

$$U_{FT}^{\dagger} U_0 U_1 U_2 U_{FT}$$

where the U_0, U_1, U_2 are single-qubit unitary operators acting on qubits 0, 1, 2. Show that this is possible (for any w) and give explicit expressions for suitable operators U_0, U_1, U_2 (depending on w) in terms of T and H, the single-qubit operators in the universal gate set $\{T, H, CNOT\}$.

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Q5 We wish to correct for single qubit phase errors of the form $|0\rangle \rightarrow e^{i\theta}|0\rangle$, $|1\rangle \rightarrow e^{-i\theta}|1\rangle$ for arbitrary real θ by realising logical qubit states $|\overline{0}\rangle$ and $|\overline{1}\rangle$ as some chosen physical n-qubit states.

5.1 Show that such a phase error on the ith physical qubit maps any state $|\psi\rangle = \alpha|\overline{0}\rangle + \beta|\overline{1}\rangle$ to a linear combination of $|\psi\rangle$ and $Z_i|\psi\rangle$.

5.2 Explain why for $n < 3$ it is impossible to correct for arbitrary single qubit phase errors.

5.3 Now consider the case $n = 3$ and choose $|\overline{0}\rangle = |+++\rangle$ and $|\overline{1}\rangle = |---\rangle$ where $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$. Show that by making suitable measurements we can choose a suitable unitary transformation to correct for any single qubit phase error. (Describe explicitly the measurement and unitary transformation operators.)

5.4 Why would the alternative choice $|\overline{0}\rangle = |000\rangle$ and $|\overline{1}\rangle = |111\rangle$ not be useful for correcting single qubit phase errors? State what type of errors could instead be corrected using this coding?

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