# EXAMINATION PAPER

| Examination Session: | Year: | Exam Code: |
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| May/June | 2020 | MATH4041-WE01 |

| Title: |
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| Partial Differential Equations IV |

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| Time (for guidance only): | 3 hours | |
| Additional Material provided: | | |
| Materials Permitted: | | |
| Calculators Permitted: | Yes | Models Permitted: There is no restriction on the model of calculator which may be used. |

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| Instructions to Candidates: | Credit will be given for your answers to all questions.<br>All questions carry the same marks.<br><br>Please start each question on a new page.<br>Please write your CIS username at the top of each page.<br><br>Show your working and explain your reasoning. |

| Revision: | |
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**Q1** Let us consider the problem

$$\begin{cases} \partial_t v + \frac{1}{2}(\partial_x v)^2 = 0, & \text{in } \mathbb{R} \times (0, +\infty), \\ v(x, 0) = v_0(x), & \text{in } \mathbb{R}, \end{cases} \tag{HJ}$$

where $v_0 : \mathbb{R} \to \mathbb{R}$ is a given function which is twice continuously differentiable on $\mathbb{R}$. We aim to solve the problem by the method of characteristics.

**1.1** Determine the type of the PDE appearing in (HJ) (i.e. is it linear; semi-linear; quasi-linear or fully nonlinear?). Justify your answer!

**1.2** Identify the Cauchy data and the Cauchy curve, and give a parametrisation of it. Use the notation $s$ for the parameter.

**1.3** We will identify the system of ODEs satisfied by the flow $\tau \mapsto (x(\tau, s), t(\tau, s))$ and the solution along the flow, $\tau \mapsto z(\tau, s) := v(x(\tau, s), t(\tau, s))$. In order to be able to solve this ODE system, one needs to introduce a new variable, $\tau \mapsto p(\tau, s) := \partial_x v(x(\tau, s), t(\tau, s))$. We rewrite the PDE in the form $\partial_t v + \partial_x v \partial_x v = \frac{1}{2}(\partial_x v)^2$ and so one equation reads as $\partial_\tau x(\tau, s) = p(\tau, s)$. Find the ODEs satisfied by $t(\tau, s)$, $z(\tau, s)$ and $p(\tau, s)$. *Hint:* to find $\partial_\tau p$, differentiate the original PDE with respect to $x$.

**1.4** Solve the new ODE system from **1.3** for $(x, t, z, p)$.

**1.5** Determine the maximal time $t_{\max} > 0$ for which the problem (HJ) has a classical solution on $\mathbb{R} \times (0, t_{\max})$. *Hint:* depending on $v_0''$, for which values of $t$ is the flow invertible?

**1.6** Find a sufficient condition on $v_0$ which allows to conclude that $t_{\max} = +\infty$ and therefore (HJ) has a global classical solution on $\mathbb{R} \times (0, +\infty)$.

**1.7** For $v_0(x) = \frac{1}{2}x^2$, find the explicit solution to (HJ). What is the value of $t_{\max}$ in this case?

**Q2** We consider Burgers' equation

$$\begin{cases} \partial_t u + \frac{1}{2}\partial_x(u^2) = 0, & \text{in } \mathbb{R} \times (0, +\infty), \\ u(x, 0) = x, & \text{in } \mathbb{R}. \end{cases} \tag{Burgers'}$$

**2.1** Can one use a theorem presented during the lectures to conclude that (Burgers') has a global classical solution on $\mathbb{R} \times (0, +\infty)$? Justify your answer!

**2.2** Show that (Burgers') has a global classical solution and find explicitly this solution.

**Q3** **3.1** Let $v \in L^1(\mathbb{R})$ and define $\tau_a v \in L^1(\mathbb{R})$ by $\tau_a v(x) = v(x-a)$, which is the translation of $v$ by $a \in \mathbb{R}$. Use a change of variables to prove that

$$\widehat{\tau_a v}(\xi) = e^{-\mathrm{i}\xi a}\widehat{v}(\xi).$$

**3.2** Let $v \in C^1(\mathbb{R})$ such that $v, v' \in L^1(\mathbb{R})$. Show that $\widehat{v'}(\xi) = \mathrm{i}\xi\widehat{v}(\xi)$.

**3.3** Let $g \in C(\mathbb{R})$. Define $G(x) := \int_0^x g(y)\,dy$. Assume that $g, G \in L^1(\mathbb{R})$. Show that $\widehat{G}(\xi) = \frac{1}{\mathrm{i}\xi}\widehat{g}(\xi)$.

**3.4** Let $f, g \in C^2(\mathbb{R})$. Use the Fourier transform and parts **3.1**, **3.2**, **3.3** to derive the solution

$$u(x,t) = \frac{1}{2}\big(f(x+t) + f(x-t)\big) + \frac{1}{2}\int_{x-t}^{x+t} g(y)\,dy \tag{1}$$

of the wave equation

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = 0, & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = f(x),\ u_t(x,0) = g(x), & x \in \mathbb{R}. \end{cases}$$

No points will be given if the solution is found with a different method. *Hint:* First take the Fourier transform in $x$ of $u$ defined in (1).

**Q4** Let $\Omega = (a_1, b_1) \times (a_2, b_2) \subset \mathbb{R}^2$ be bounded. Let $u : \overline{\Omega} \times [0, \infty) \to \mathbb{R}$ be a smooth function satisfying

$$\begin{cases} u_t(\mathbf{x}, t) - k\Delta u(\mathbf{x}, t) = f(\mathbf{x}), & (\mathbf{x}, t) \in \Omega \times (0, \infty), \\ u(\mathbf{x}, t) = g(\mathbf{x}), & \mathbf{x} \in \partial\Omega \times [0, \infty), \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}), & \mathbf{x} \in \Omega, \end{cases}$$

where $u_0, f, g$ are smooth functions and $k > 0$. Let $v : \overline{\Omega} \to \mathbb{R}$ be a smooth, time independent solution of the equation

$$\begin{cases} -k\Delta v(\mathbf{x}) = f(\mathbf{x}), & \mathbf{x} \in \Omega, \\ v(\mathbf{x}) = g(\mathbf{x}), & \mathbf{x} \in \partial\Omega. \end{cases}$$

Define the difference $w(\mathbf{x}, t) = u(\mathbf{x}, t) - v(\mathbf{x})$.

**4.1** Check that $w$ satisfies

$$\begin{cases} w_t(\mathbf{x}, t) - k\Delta w(\mathbf{x}, t) = 0, & (\mathbf{x}, t) \in \Omega \times (0, \infty), \\ w(\mathbf{x}, t) = 0, & \mathbf{x} \in \partial\Omega \times [0, \infty), \\ w(\mathbf{x}, 0) = u_0(\mathbf{x}) - v(x), & \mathbf{x} \in \Omega. \end{cases} \tag{2}$$

**4.2** Show that $w$ satisfies the following Poincaré inequality on the rectangle $\Omega$: There exists $C_\Omega > 0$ (independent of $u, v$) such that

$$\forall\, t \in (0, \infty) : \quad \int_\Omega w(\mathbf{x}, t)^2 \, d\mathbf{x} \le C_\Omega \int_\Omega |\nabla w(\mathbf{x}, t)|^2 \, d\mathbf{x}. \tag{3}$$

*Hint:* You may use without proof the following one dimensional Poincaré inequality: There exists $C_{a,b} > 0$ such that every function $h \in C^1([a, b])$ with $h(a) = h(b) = 0$ satisfies $\int_a^b h^2 \, dx \le C_{a,b} \int_a^b (h')^2 \, dx$.

**4.3** Define $E(t) = \int_\Omega w(\mathbf{x}, t)^2 \, d\mathbf{x}$. Use (2) and (3) to prove that there exists a constant $C > 0$ (independent of $u, v$) such that

$$\forall\, t \in (0, \infty) : \quad E'(t) \le -CE(t).$$

**4.4** State Grönwall's inequality (without proof) and use it to prove that

$$\lim_{t \to \infty} \int_\Omega (u(\mathbf{x}, t) - v(\mathbf{x}))^2 \, d\mathbf{x} = 0,$$

which means that the solution $u$ converges to the time independent solution $v$ (with respect to the $L^2(\Omega)$ norm).

**Q5** Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function [i.e. $f((1-t)x + ty) \leq (1-t)f(x) + tf(y)$ for all $t \in [0,1]$ and for all $x, y \in \mathbb{R}$] which is twice continuously differentiable on $\mathbb{R} \setminus \{0\}$ and is not differentiable at 0.

**5.1** Let $x, y, z \in \mathbb{R}$ such that $x < z < y$. Show that

$$\frac{f(x) - f(z)}{x - z} \leq \frac{f(y) - f(x)}{y - x}.$$

**5.2** We know that $f'(0-)$ and $f'(0+)$, the left and right derivatives at zero, exist. Show that $f'(0-) < f'(0+)$. *Hint:* use **5.1**.

**5.3** Show that there exists $\alpha > 0$ such that $f''(0) = \alpha\delta_0$ (here $f''(0)$ stands for the distributional second derivative of $f$ restricted to the set $\{0\}$ and $\delta_0$ stands for the Dirac delta at 0). Determine the value of $\alpha$.

**5.4** Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = |x|$. Verify that this function satisfies the assumptions of this question. Compute $f'(0-)$, $f'(0+)$ and $f''(0)$.