Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2020	MATH4051-WE01

Title:

General Relativity IV

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. Show your working and explain your reasoning.

Revision:	

ED01/2020

Q1 **1.1** A two-dimensional space has coordinates $x^\mu = (r, \theta)$. New local coordinates $\tilde{x}^\mu = (t, x)$ are defined by $t = r\sinh\theta$ and $x = r\cosh\theta$.

(i) A vector field V^μ has components $V^\mu = (-\sinh\theta, r^{-1}\cosh\theta)$ with respect to the original coordinates x^μ. What are its components $\widetilde{V}^\mu$ with respect to the new coordinates?

(ii) A $[1,1]$ tensor field $T^\mu_{\ \nu}$ has $T^r_{\ r} = 1$, all other components vanishing with respect to the original coordinates x^μ. What are its components $\widetilde{T}^\mu_{\ \nu}$ with respect to the new coordinates $\tilde{x}^\mu$?

1.2 This part of this question concerns Killing vector fields.

(i) State the condition (in any of its forms) for a vector field V_μ to be a Killing vector field.

(ii) Consider flat space in 2d Cartesian coordinates: $ds^2 = dx^2 + dy^2$. Write down all of the Killing vector fields for this space.

(iii) A *conformal* Killing vector field is a vector field K_μ that satisfies the following equation:

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = f(x) g_{\mu\nu}$$

where $f(x)$ is any function on spacetime. Show that the vector field $K^\mu = x^\mu$ is a conformal Killing vector field for 2d flat space, for a particular choice of $f(x)$ that you should determine.

Q2 **2.1** In a space-time with metric $ds^2 = -2x dt^2 + dx^2 + 2dxdy + 2dy^2 + dz^2$, a curve is defined by $t(s) = s, x(s) = s^2, y(s) = s, z(s) = 1$.

(i) Compute the norm of the tangent vector.

(ii) Is the curve timelike, null or spacelike?

(iii) Find an affine parameter for the curve.

2.2 Consider the spacetime with coordinates $x^\mu = (t, z)$ and metric

$$ds^2 = z^{-2}(-dt^2 + dz^2).$$

Compute the Christoffel symbols $\Gamma^\mu_{\nu\lambda}$, and determine the components of Killing's equation in this spacetime. Show that $U^\mu = (1, 0)$ and $V^\mu = (t, z)$ are Killing vectors for this metric.

Q3 The Riemann curvature tensor is defined by $[\nabla_\mu, \nabla_\nu]V^\lambda = R^\lambda_{\ \rho\mu\nu} V^\rho$.

3.1 Show that the commutator $[\nabla_\mu, \nabla_\nu]$ satisfies

$$[\nabla_\mu, \nabla_\nu](V^\lambda \omega_\lambda) = \omega_\lambda [\nabla_\mu, \nabla_\nu] V^\lambda + V^\lambda [\nabla_\mu, \nabla_\nu] \omega_\lambda.$$

Use this to derive an expression for $[\nabla_\mu, \nabla_\nu]\omega_\lambda$.

3.2 Similarly derive an expression for $[\nabla_\mu, \nabla_\nu] T^{\rho\sigma}$.

3.3 Suppose that K^μ is a Killing vector. Show that this implies $\nabla_\mu \nabla_\sigma K^\mu = R_{\lambda\sigma} K^\lambda$ and $\nabla_\sigma \nabla_\mu \nabla^\sigma K^\mu = 0$.

Q4 Consider the metric

$$ds^2 = -\left(1 - \frac{r^2}{L^2}\right) dt^2 + \frac{dr^2}{1 - \frac{r^2}{L^2}} + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

where L is a real constant.

4.1 Identify the coordinate and curvature singularities of this metric, if any.

4.2 Use conserved quantities to reduce the massive geodesic equation in the above spacetime to the form

$$\frac{1}{2}\left(\frac{dr}{ds}\right)^2 + V(r) = E,$$

where s is an affine parameter along the geodesic. You should express $V(r)$ and E in terms of constants of the motion. You may assume that the particle motion happens in the equatorial plane $\theta = \frac{\pi}{2}$.

4.3 Describe in words the possible trajectories for freely-falling particles living in this spacetime. Is a circular orbit possible? If so, state the radial position of the circular orbit in terms of constants of the motion. If not, explain why.

4.4 Consider a particle released from rest at $r = r_0$, $0 < r_0 < L$. Compute how much time elapses along the particle's worldline until it reaches $r = L$. You may leave your answer in the form of a definite integral if one arises.

Q5 The Friedmann equation for cosmological dynamics with no matter but with a cosmological constant Λ is

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2} = \frac{\Lambda}{3} \tag{1}$$

where κ denotes the spatial curvature, $a(t)$ is the scale factor and an overdot denotes the derivative with respect to t.

5.1 Show that the following metric with positive spatial curvature satisfies the Friedmann equation

$$ds^2 = -dt^2 + L^2 \cosh^2\left(\frac{t}{L}\right)\left(d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\right) \tag{2}$$

for an appropriate choice of the constant L which you should determine.

5.2 At $t = t_0$, a light ray is emitted at $\chi = 0$ and moves only in the χ direction. As a function of t_0, how far does it move in χ by the time the universe ends at $t = +\infty$?

5.3 Consider two stationary astronauts: Astronaut A is at $\chi = 0$ and Astronaut B is at $\chi = \chi_B$ (and fixed θ, ϕ). Use the fact that the Christoffel symbols on this metric satisfy $\Gamma^\mu_{tt} = 0$ for all choices of the coordinate μ to show that these trajectories are geodesics.

5.4 Astronaut A emits a photon in the χ direction at $t = 0$ with frequency ω_A. This light ray is observed by Astronaut B. Find, as a function of χ_B and ω_A, the time t_B at which the light ray reaches Astronaut B and the frequency ω_B observed there.

The following indefinite integral may be useful for this problem: $\int dx\ \mathrm{sech}(x) = 2\arctan\left(\tanh\left(\frac{x}{2}\right)\right) + C$.