Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2020	MATH4061-WE01

Title:

Advanced Quantum Theory IV

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. Show your working and explain your reasoning.

Revision:	

SECTION A

Q1 The action for two real scalar fields $\varphi_1(x)$ and $\varphi_2(x)$ is given by

$$S = \int \mathrm{d}^4x \left\{ -\frac{1}{2}m^2\varphi_1(x)^2 - \frac{1}{2}m^2\varphi_2(x)^2 - \frac{1}{2}\partial_\mu\varphi_1(x)\partial^\mu\varphi_1(x) - \frac{1}{2}\partial_\mu\varphi_2(x)\partial^\mu\varphi_2(x) - \lambda\Big(\varphi_1^2(x) + \varphi_2^2(x)\Big)^5 \right\}.$$

1.1 Write the equations of motion for the fields $\varphi_1(x)$ and $\varphi_2(x)$.

1.2 Write all global, continuous symmetries which leave this action invariant. Show explicitly or argue that the action is invariant under these symmetries and derive the Noether current(s) associated with these symmetries.

The Virasoro constraints for the quantum open relativistic string are given by

$$\hat{L}_m := \frac{1}{2}\left(\sum_{n=-\infty}^{\infty} :\hat{\alpha}^\mu_{m-n}\hat{\alpha}^\nu_n: \eta_{\mu\nu}\right) - a\delta_{m,0} = 0\,,$$

with $\hat{\alpha}^\mu_0 = \sqrt{2\alpha'}\,\hat{p}^\mu$.

1.3 Describe briefly the origin of this constraint.

1.4 The number operator is defined as

$$\hat{N} := \sum_{n=1}^{\infty} \hat{\alpha}^\mu_{-n}\hat{\alpha}^\nu_n\eta_{\mu\nu}\ .$$

Derive a formula for the mass of a string state in terms of $\hat{N}$.

1.5 $\hat{\alpha}^\mu_m$ satisfies the commutation relations

$$[\hat{\alpha}^\mu_m, \hat{\alpha}^\nu_n] = m\delta_{m+n}\eta^{\mu\nu}\ .$$

Compute the commutator $[\hat{N}, \hat{\alpha}^\mu_m]$.

1.6 Using light cone gauge, define the vacuum state and the first excited state. Describe physically what they correspond to and explain how this fixes the value of a.

Q2 An action for two free, real scalar fields $\varphi_1(x)$ and $\varphi_2(x)$ is given by

$$S = \int \mathrm{d}^4x \left(-\frac{1}{2}m_1^2\varphi_1^2(x) - \frac{1}{2}\partial_\mu\varphi_1(x)\partial^\mu\varphi_1(x) - \frac{1}{2}m_2^2\varphi_2^2(x) - \frac{1}{2}\partial_\mu\varphi_2(x)\partial^\mu\varphi_2(x) \right)$$

$\mu = 0, 1, 2, 3.$

2.1 Write down the quantum version of the general solution for the fields φ_1 and φ_2 in terms of creation and annihilation operators. Write down the basic commutation relations between creation and annihilation operators.

2.2 Write down the quantum state which consists of two particles of type φ_1 with momenta $\mathbf{p_1}$ and $\mathbf{p_2}$ and two particles of type φ_2 with momenta $\mathbf{p_3}$ and $\mathbf{p_4}$. Using the normal ordered quantum Hamiltonian $\hat{H}$

$$\hat{H} = \int \frac{d^3p}{(2\pi)^3} \left(\omega_p^{(1)} \hat{a}^\dagger_{\boldsymbol{p}} \hat{a}_{\boldsymbol{p}} + \omega_p^{(2)} \hat{b}^\dagger_{\boldsymbol{p}} \hat{b}_{\boldsymbol{p}} \right) \qquad \omega_p^{(i)} = \sqrt{\boldsymbol{p}^2 + m_i^2} \qquad (i = 1, 2)$$

for the action (1), explicitly compute the energy of this state. In the expression for $\hat{H}$, the operators $\hat{a}^\dagger_{\boldsymbol{p}}, \hat{a}_{\boldsymbol{p}}, \hat{b}^\dagger_{\boldsymbol{p}}, \hat{b}_{\boldsymbol{p}}$ are creation and annihilation operators for the fields φ_1 and φ_2.

2.3 Based on your understanding of the structure of normal ordered quantum Hamiltonians write down the normal ordered expression for the conserved momentum $\hat{P}_i$ for this system and motivate your answer.

2.4 By explicit computation determine the commutator between two normal ordered operators $\hat{P}_i$ and $\hat{P}_j$ and explain why such a result is expected.

Q3 The action for four real scalar fields $\varphi_1(x), \varphi_2(x), \varphi_3(x), \varphi_4(x)$ is given by

$$S = -\int \mathrm{d}^4x \left\{ \frac{1}{2} \left(\sum_{i=1}^{4} \partial_\mu\varphi_i(x)\partial^\mu\varphi_i(x) + \sum_{i=1}^{4} m_i^2\varphi_i(x)^2 \right) + \lambda\varphi_1(x)\varphi_2(x)\varphi_3(x)\varphi_4(x) \right\}$$

where λ is a real number, a coupling constant.

3.1 Write down the Feynman rules for this theory in position and momentum space. Write down the integral expression for the Feynman propagators.

3.2 List all the vacuum bubbles which appear in this theory up to and including order λ^2. You should draw all the graphs and write the expressions for these graphs in position space. You do not need to evaluate any of the graphs.

3.3 Explain whether it is possible in this theory for two particles of type "i" which originate from the field φ_i, to scatter and produce two particles of type "j" which originate from a field φ_j, where $i \neq j$. If this is not possible explain why, and if it is, justify your answer by sketching at least one graph which would contribute to such scattering.

3.4 Evaluate the four-point correlators

$$\langle\Omega|T\{\varphi_1(x)\varphi_1(y)\varphi_1(z)\varphi_1(w)\}|\Omega\rangle \quad \text{and} \quad \langle\Omega|T\{\varphi_1(x)\varphi_2(y)\varphi_3(x)\varphi_4(y)\}|\Omega\rangle\,,$$

up to and including second order in perturbation theory.

Q4 Consider a 0-dimensional "field theory" with action

$$S = -\frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 \, .$$

4.1 What does the expression

$$I_\lambda = \frac{\displaystyle\int_{-\infty}^{\infty} \phi^2 \, e^{\frac{i}{\hbar}S} \, \mathrm{d}\phi}{\displaystyle\int_{-\infty}^{\infty} e^{\frac{i}{\hbar}S} \mathrm{d}\phi}$$

represent physically?

4.2 The Fresnel integral is given as

$$\int e^{-ia\phi^2} d\phi = \sqrt{\frac{\pi}{ia}}$$

(which you can assume without proof).

Use this to compute I_0.

Hint: Differentiate both sides of the Fresnel integral with respect to a to obtain expressions for more general integrals.

4.3 Similarly compute the order λ term to the series expansion of I_λ.

4.4 Write down the Feynman diagrams corresponding to the above expressions and explain how they give the same results.

[For the Feynman rules: the vertex is given by $-i\lambda/\hbar$ and you can deduce the propagator from your result for I_0].

4.5 What is the two-particle "scattering amplitude" to $O(\lambda)$?

4.6 Thus give the physical mass in this theory to $O(\lambda)$.

Q5 Consider the closed string action,

$$S = -\frac{T}{2}\int \mathrm{d}\tau \int_0^{2\pi} \mathrm{d}\sigma \sqrt{-h}h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu \,.$$

5.1 Derive the equations of motion and the constraints from the above Lagrangian.
Hint: You may use without proof that $\frac{\partial}{\partial h^{\alpha\beta}}\sqrt{-h} = -\frac{1}{2}h_{\alpha\beta}\sqrt{-h}$.

5.2 Use the constraints to find $h_{\alpha\beta}$ in terms of the derivatives of the embedding coordinates (up to an undetermined scale) and insert the solution back into the action, what do you obtain? Why is this undetermined scale present?

5.3 Show that the following is a solution to the equations of motion and constraints:

$$h_{\alpha\beta} = \begin{pmatrix} -a^2\tau^2 & 0 \\ 0 & 1 \end{pmatrix}\,, \qquad \begin{aligned} X^0 &= R\tau^2\,, \\ X^1 &= R\cos(\sigma)\cos\tau^2\,, \\ X^2 &= R\cos(\sigma)\sin\tau^2 \end{aligned}$$

for some constant a you should find.

5.4 The Polyakov action for a p-brane (an object with p space-like directions and one time-like direction embedded in D-dimensional space-time) reads

$$S = \frac{1}{2}\int \Big(-\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu + (p-1)\sqrt{-h}\Big)\,\mathrm{d}^{p+1}\sigma\,.$$

Here we have collectively denoted the $p+1$ world-volume directions by σ. The worldsheet indices α, β run from $0, 1, \ldots, p$, whereas the space-time indices $\mu, \nu = 0, 1, \ldots D-1$. $X^\mu(\sigma)$ are the embedding coordinates.

For $p > 1$, compute the equation of motion for $h_{\alpha\beta}$ (constraints) in this case and use it to completely determine $h_{\alpha\beta}$ in terms of the derivatives of the embedding coordinates. Insert the solution back into the action to obtain the Nambu-Goto action for p-branes.

Hint: You may still use without proof that $\frac{\partial}{\partial h^{\alpha\beta}}\sqrt{-h} = -\frac{1}{2}h_{\alpha\beta}\sqrt{-h}$.

5.5 Comment on any differences with the string case.