Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2020	MATH4091-WE01

Title:
Stochastic Processes IV

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. Show your working and explain your reasoning.

Revision:	

ED01/2020

Q1 Let $X_1, X_2, \ldots$ be independent identically distributed random variables with $\mathsf{P}(X_n = 1) = p = 1 - \mathsf{P}(X_n = -1)$ for some $p \in [1/2, 1)$. Set $S_0 = 0$ and $S_n = \sum_{i=1}^n X_i$. For $x \in \mathbb{Z}$ define $T_x := \min\{n \geq 0 : S_n = x\}$ and suppose a, b are integers with $a < 0 < b$.

1.1 Show that $T_a \wedge T_b := \min(T_a, T_b)$ is a stopping time and prove that $\mathsf{E}(T_a \wedge T_b)$ is finite.

1.2 Prove that $(S_n)_{n\geq 0}$ is a martingale if and only if $p = 1/2$, and deduce that $\mathsf{P}(T_a < T_b) = b/(b-a)$ when $p = 1/2$.

1.3 Now suppose $p > 1/2$ and let $Y_n = \beta^{S_n}$ for all $n \geq 0$. Find the value of $\beta \in (0, 1)$ such that $(Y_n)_{n\geq 0}$ is a martingale with respect to $(S_n)_{n\geq 0}$.

1.4 Hence calculate $\mathsf{P}(T_a < T_b)$ in terms of $p > 1/2$, a, and b.

You should show all your working and justify your calculations with suitable explanation.

Q2 Let $(X_n)_{n\geq 0}$, $X_0 = x$, be a random walk on the complete graph K_m on $m > 2$ vertices, such that at every step it jumps to any of the other $m-1$ vertices uniformly at random. Use an appropriate coupling to show that for every vertex $v \in K_m$ we have $|\mathsf{P}(X_n = v) - \frac{1}{m}| \leq \mathrm{e}^{-an}$ for some $a = a_m > 0$. You should show all your working and justify your calculations with suitable explanation.

Q3 Let $X(t)$, $t \geq 0$, be a continuous-time Markov chain on the state space $\{1, 2, 3\}$ whose generator (Q-matrix) is

$$Q = \begin{pmatrix} -6 & 4 & 2 \\ 0 & -2 & 2 \\ 4 & 4 & -8 \end{pmatrix}.$$

3.1 Write down the backward Kolmogorov equations for the transition probabilities $p_{ij}(t)$, i, $j \in \{1, 2, 3\}$.

3.2 Show that

$$P(t) = \exp\{tQ\} = \sum_{k\geq 0} \frac{t^k}{k!} Q^k$$

is a unique solution to the equations you obtained in question **3.1**.

3.3 Define the resolvent $R(\lambda)$ for $X(t)$ and find

$$p_{31}(t) = \mathsf{P}\big(X(t) = 1 \mid X(0) = 3\big).$$

You should show all your working and justify your calculations with suitable explanation.

Q4 **4.1** Carefully define a Brownian motion, state its Markov and strong Markov properties.

4.2 State the stopping theorem for bounded martingales.

Let $(B_t)_{t\geq 0}$ be a Brownian motion starting at the origin ($B_0 = 0$).

4.3 Carefully show that for all $0 \leq s < t$ we have

$$\mathsf{E}\big((B_t)^3 - 3tB_t | B_r, 0 \leq r \leq s\big) = (B_s)^3 - 3sB_s\,,$$
$$\mathsf{E}\big(e^{\theta B_t - t\theta^2/2} | B_r, 0 \leq r \leq s\big) = e^{\theta B_s - s\theta^2/2}\,,$$

where θ is a real number, i.e., that $(B_t)^3 - 3tB_t$ and $e^{\theta B_t - t\theta^2/2}$ are martingales with respect to the natural filtration.

4.4 For $a > 0$, let $\tau = \inf\big\{t \geq 0 : |B_t| \geq a\big\}$ be the exit time from the interval $(-a, a)$. Show that $\mathsf{E}(e^{-\beta\tau}) = 2/\big(e^{a\sqrt{2\beta}} + e^{-a\sqrt{2\beta}}\big)$ for all $\beta \geq 0$.

You should show all your working and justify your calculations with suitable explanation.

Q5 **5.1** Carefully define a renewal process.

5.2 Let $N(t)$ be a renewal process with interarrival times $(T_i)_{i\geq 1}$. Suppose that the i-th renewal earns a reward R_i, so that the total reward at time t equals $\mathcal{R}(t) = \sum_{i=1}^{N(t)} R_i$. Assuming that the pairs (T_i, R_i) are independent and identically distributed, with $0 < \mathsf{E}(T_1) < \infty$ and $\mathsf{E}(R_1) < \infty$, show that, with probability one,

$$\frac{\mathcal{R}(t)}{t} \to \frac{\mathsf{E}(R_1)}{\mathsf{E}(T_1)} \quad \text{as } t \to \infty.$$

5.3 Two city car rental companies operate alternative pricing schemes. Company A charges an hourly rate of £4 per hour for the first 2 hours, rising to £5 per hour for all subsequent hours. (Assume that any fraction of an hour is charged pro-rata; for example, a rental time of 3.2 hours will cost £14.) Company B charges a flat fee of £5 for up to 3 hours, but has an additional penalty charge of £15 if the car is not returned within the 3-hour period.

A driver (wishing to make repeated use of a hire car) estimates that the length of time in hours of each rental will be independent and identically distributed uniformly on $[0, 4]$. Which company will work out cheaper for the driver in the long-run?

You should show all your working and justify your calculations with suitable explanation.