

EXAMINATION PAPER

Examination Session: May/June

2020

Year:

Exam Code:

MATH4091-WE01

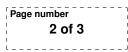
Title:

Stochastic Processes IV

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks.
	Please start each question on a new page. Please write your CIS username at the top of each page.
	Show your working and explain your reasoning.

Revision:



- **Q1** Let X_1, X_2, \ldots be independent identically distributed random variables with $\mathsf{P}(X_n = 1) = p = 1 \mathsf{P}(X_n = -1)$ for some $p \in [1/2, 1)$. Set $S_0 = 0$ and $S_n = \sum_{i=1}^n X_i$. For $x \in \mathbb{Z}$ define $T_x := \min\{n \ge 0 : S_n = x\}$ and suppose a, b are integers with a < 0 < b.
 - **1.1** Show that $T_a \wedge T_b := \min(T_a, T_b)$ is a stopping time and prove that $\mathsf{E}(T_a \wedge T_b)$ is finite.
 - **1.2** Prove that $(S_n)_{n\geq 0}$ is a martingale if and only if p = 1/2, and deduce that $\mathsf{P}(T_a < T_b) = b/(b-a)$ when p = 1/2.
 - **1.3** Now suppose p > 1/2 and let $Y_n = \beta^{S_n}$ for all $n \ge 0$. Find the value of $\beta \in (0,1)$ such that $(Y_n)_{n\ge 0}$ is a martingale with respect to $(S_n)_{n\ge 0}$.
 - **1.4** Hence calculate $P(T_a < T_b)$ in terms of p > 1/2, a, and b.

You should show all your working and justify your calculations with suitable explanation.

- **Q2** Let $(X_n)_{n\geq 0}$, $X_0 = x$, be a random walk on the complete graph K_m on m > 2 vertices, such that at every step it jumps to any of the other m-1 vertices uniformly at random. Use an appropriate coupling to show that for every vertex $v \in K_m$ we have $|\mathsf{P}(X_n = v) \frac{1}{m}| \leq e^{-an}$ for some $a = a_m > 0$. You should show all your working and justify your calculations with suitable explanation.
- **Q3** Let X(t), $t \ge 0$, be a continuous-time Markov chain on the state space $\{1, 2, 3\}$ whose generator (*Q*-matrix) is

$$Q = \begin{pmatrix} -6 & 4 & 2\\ 0 & -2 & 2\\ 4 & 4 & -8 \end{pmatrix}.$$

- **3.1** Write down the backward Kolmogorov equations for the transition probabilities $p_{ij}(t), i, j \in \{1, 2, 3\}.$
- 3.2 Show that

$$P(t) = \exp\{tQ\} = \sum_{k \ge 0} \frac{t^k}{k!} Q^k$$

is a unique solution to the equations you obtained in question **3.1**.

3.3 Define the resolvent $R(\lambda)$ for X(t) and find

$$p_{31}(t) = \mathsf{P}(X(t) = 1 \mid X(0) = 3).$$

You should show all your working and justify your calculations with suitable explanation.

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- **Q4 4.1** Carefully define a Brownian motion, state its Markov and strong Markov properties.
 - 4.2 State the stopping theorem for bounded martingales.
 - Let $(B_t)_{t\geq 0}$ be a Brownian motion starting at the origin $(B_0 = 0)$.
 - **4.3** Carefully show that for all $0 \le s < t$ we have

$$\mathsf{E}((B_t)^3 - 3tB_t | B_r, 0 \le r \le s) = (B_s)^3 - 3sB_s,$$
$$\mathsf{E}(e^{\theta B_t - t\theta^2/2} | B_r, 0 \le r \le s) = e^{\theta B_s - s\theta^2/2},$$

where θ is a real number, i.e., that $(B_t)^3 - 3tB_t$ and $e^{\theta B_t - t\theta^2/2}$ are martingales with respect to the natural filtration.

4.4 For a > 0, let $\tau = \inf\{t \ge 0 : |B_t| \ge a\}$ be the exit time from the interval (-a, a). Show that $\mathsf{E}(e^{-\beta\tau}) = 2/(e^{a\sqrt{2\beta}} + e^{-a\sqrt{2\beta}})$ for all $\beta \ge 0$.

You should show all your working and justify your calculations with suitable explanation.

- Q5 5.1 Carefully define a renewal process.
 - **5.2** Let N(t) be a renewal process with interarrival times $(T_i)_{i\geq 1}$. Suppose that the *i*-th renewal earns a reward R_i , so that the total reward at time *t* equals $\mathcal{R}(t) = \sum_{i=1}^{N(t)} R_i$. Assuming that the pairs (T_i, R_i) are independent and identically distributed, with $0 < \mathsf{E}(T_1) < \infty$ and $\mathsf{E}(R_1) < \infty$, show that, with probability one,

$$\frac{\mathcal{R}(t)}{t} \to \frac{\mathsf{E}(R_1)}{\mathsf{E}(T_1)} \quad \text{as } t \to \infty.$$

5.3 Two city car rental companies operate alternative pricing schemes. Company A charges an hourly rate of £4 per hour for the first 2 hours, rising to £5 per hour for all subsequent hours. (Assume that any fraction of an hour is charged pro-rata; for example, a rental time of 3.2 hours will cost £14.) Company B charges a flat fee of £5 for up to 3 hours, but has an additional penalty charge of £15 if the car is not returned within the 3-hour period.

A driver (wishing to make repeated use of a hire car) estimates that the length of time in hours of each rental will be independent and identically distributed uniformly on [0, 4]. Which company will work out cheaper for the driver in the long-run?

You should show all your working and justify your calculations with suitable explanation.