Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2020	MATH4161-WE01

Title: Algebraic Topology IV

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. Show your working and explain your reasoning.

Revision:	

ED01/2020

Q1 **1.1** State the Mayer-Vietoris sequence theorem.

1.2 Compute the homology of the topological space $S^1 \times S^2$.

1.3 Prove that $\mathbb{CP}^2$ is not a retract of $\mathbb{CP}^3$. Recall: given a subspace $i\colon X \hookrightarrow Y$, a retraction is a map $r\colon Y \to X$ with $r \circ i = \mathrm{Id}$. Hint: first compute the cohomology ring of $\mathbb{CP}^n$ for $n = 2, 3$.

Q2 **2.1** Define a good pair (X, A).

2.2 Let X be a topological space. The cone on X, $\mathscr{C}(X)$, is defined to be the quotient space

$$\frac{X \times I}{X \times \{0\}}.$$

Show that $(X \times I, X \times \{0\})$ is a good pair and compute the homology groups of $\mathscr{C}(X)$.

2.3 Compute the reduced homology $\widetilde{H}_*(SX)$, where $SX := \mathscr{C}(X) \cup_{X\times\{1\}} \mathscr{C}(X)$ is the suspension of X, in terms of the homology groups of X.

Q3 True or false? For each item, either prove or disprove.

3.1 Two topological spaces with the same homology groups are homeomorphic.

3.2 If $0 \to \mathbb{Z} \to G \to \mathbb{Z}/6 \to 0$ is an exact sequence of abelian groups, then $G \cong \mathbb{Z} \oplus \mathbb{Z}/6$.

3.3 For every topological space X, X is homotopy equivalent to $X \times \mathbb{R}$.

3.4 There exists a CW-complex X with homology groups

$$H_0(X) \cong \mathbb{Z} \cong H_3(X) \text{ and } H_1(X) \cong \mathbb{Z}/4 \cong H_2(X)$$

and $H_k(X) = 0$ for $k \geq 4$.

3.5 There exists a closed, orientable manifold M with homology

$$H_0(M) \cong \mathbb{Z} \cong H_3(M) \text{ and } H_1(M) \cong \mathbb{Z}/4 \cong H_2(M)$$

and $H_k(M) = 0$ for $k \geq 4$.

Q4 **4.1** Show that the short exact sequence

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$$

induces, for every topological space X, a short exact sequence of cochain complexes

$$0 \to C^*(X;\mathbb{Z}/2) \to C^*(X;\mathbb{Z}/4) \to C^*(X;\mathbb{Z}/2) \to 0.$$

4.2 Let

$$\beta\colon H^n(X;\mathbb{Z}/2) \to H^{n+1}(X;\mathbb{Z}/2)$$

be the connecting homomorphism in the associated long exact sequence. Write down this long exact sequence for $X = \mathbb{RP}^2$. Compute the map

$$\beta\colon H^1(\mathbb{RP}^2;\mathbb{Z}/2) \to H^2(\mathbb{RP}^2;\mathbb{Z}/2).$$

(You should first compute $H^i(\mathbb{RP}^2;\mathbb{Z}/2)$ for every i.)

4.3 Prove that $\beta(x) = x \smile x$ for every $x \in H^1(\mathbb{RP}^2;\mathbb{Z}/2)$. You may quote $\mathbb{Z}/2$-coefficient Poincaré duality if you use it.

Q5 **5.1** Define the Euler characteristic $\chi(X)$ of a finite CW-complex X.

5.2 Prove that $\chi(X)$ is independent of the choice of CW structure on X, stating carefully any results from the course that you use. You may assume that for

$$0 \to A \to B \to C \to 0$$

a short exact sequence of finitely generated abelian groups, the ranks satisfy $\mathrm{rk}(B) = \mathrm{rk}(A) + \mathrm{rk}(C)$.

5.3 Prove that the Euler characteristic of a closed, orientable 3-manifold is zero.

5.4 Let X be a finite CW-complex and let $p\colon \widetilde{X} \to X$ be a k-sheeted covering space, for $0 < k < \infty$. Prove that $\chi(\widetilde{X}) = k \cdot \chi(X)$.

5.5 Let Σ_g be a closed, connected, orientable surface of genus g. Use the previous question to place constraints on the values of h (in terms of g) for which there exists a connected covering space $p\colon \widetilde{\Sigma}_g \to \Sigma_g$ with $\widetilde{\Sigma}_g \cong \Sigma_h$.

5.6 Show that all your permitted values of h are realised by covering maps $p\colon \Sigma_h \to \Sigma_g$.