

EXAMINATION PAPER

Examination Session: May/June

Year: 2020

Exam Code:

MATH4171-WE01

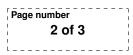
Title:

Riemannian Geometry IV

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks.
	Please start each question on a new page. Please write your CIS username at the top of each page.
	Show your working and explain your reasoning.

Revision:



- Q1 1.1 For given $a \in \mathbb{R}$ denote $M_a = \{(x, y, z) \in \mathbb{R}^3 \mid (x^2 y^2 + z + 1)^2 = a^2\}.$
 - (i) Find all $a \in \mathbb{R}$ such that M_a is a 2-dimensional smooth manifold.
 - (ii) For a = 1, find two curves $\gamma_i : [0,1] \to M_1$, i = 1,2 such that $\gamma_1(0) = \gamma_2(0) = (0,0,0)$ and $\{\gamma'_1(0), \gamma'_2(0)\}$ is a basis for $T_{(0,0,0)}M_1$.

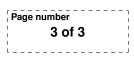
1.2 Let X, Y be two vector fields on \mathbb{R}^3 given by

$$X(x,y,z) = (x+y)\frac{\partial}{\partial x} - yz\frac{\partial}{\partial z}, \qquad Y(x,y,z) = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}.$$

- (i) Compute the Lie bracket [X, Y] of X and Y.
- (ii) Let $M = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\}$ be a coordinate plane. Show that the restriction of [X, Y] to M is a vector field on M.
- **Q2** A geodesic $c : [0, \infty) \to M$ in a Riemannian manifold M is called a *ray starting* from c(0) if it minimizes the distance between c(0) and c(t) for any $t \in (0, \infty)$.
 - **2.1** Define the Ricci curvature Ric(v) on a Riemannian manifold. State the Bonnet Myers theorem.
 - **2.2** Let M be a complete connected Riemannian manifold. Suppose that there exists an $\varepsilon > 0$ such that $Ric(v) > \varepsilon$ for each $v \in \{v \in TM \mid ||v|| = 1\}$. Show that M contains no ray starting from any point $p \in M$.
 - **2.3** Find an example of a complete Riemannian manifold N of positive Ricci curvature containing a ray starting from p for every point $p \in N$. (You do **not** need to prove that the curvature of N is positive and you do **not** need to prove completeness of N, but you **need** to show an existence of a ray for every point).
- **Q3** Consider the set F of functions from \mathbb{C} to \mathbb{C} given by

$$F = \{ f_{a,b} : \mathbb{C} \to \mathbb{C} \mid f_{a,b}(z) = az + b, \ a, b \in \mathbb{C}, \ a \neq 0 \}.$$

- **3.1** Show that all functions belonging to F form a group G with composition as the group operation.
- **3.2** Define the notion of a Lie group and show that the group G defined in (a) is a Lie group. Find the dimension of G.
- **3.3** Given a vector $v \in T_e G$ (where $e \in G$ is the neutral element), find the left-invariant vector field X on G such that X(e) = v.
- **3.4** Let g be the left-invariant metric on G which at the neutral element f(z) = z coincides with the Euclidean metric $(g_{ij} = I)$. Find the coefficients of g at any point $h = f_{a_0,b_0} \in G$.





Q4 4.1 Let ∇ be an affine connection on $M = \mathbb{R}^3$ defined by

$$\Gamma^3_{12}=\Gamma^1_{23}=\Gamma^2_{31}=\Gamma^3_{21}=\Gamma^1_{32}=\Gamma^2_{13}=1,$$

with all the other Christoffel symbols being zero. Show that this connection is torsion-free.

- **4.2** Show that the connection defined in (**4.1**) does not have the Riemannian property.
- 4.3 State the theorem of Hopf Rinow.
- **4.4** A Riemannian manifold (M, g) is called *homogeneous* if, for any pair $p, q \in M$, there exists an isometry $f : M \to M$ such that f(p) = q. Show that any homogeneous manifold must be complete.
- **Q5** Let $\mathbb{H}^3 = \{(x, y, z) \in \mathbb{R}^3 | z > 0\}$ be 3-dimensional hyperbolic space (the model in the upper halfspace), with metric \tilde{g} given by

$$\tilde{g}_{ii} = 1/z^2$$
, $\tilde{g}_{ij} = 0$ if $i \neq j$.

Consider $M \subset \mathbb{H}^3$ parametrized by (φ, θ) via

$$(x, y, z) = ((1 + \cos \theta) \cos \varphi, (1 + \cos \theta) \sin \varphi, \sin \theta), \quad \varphi, \theta \in (0, \pi).$$

5.1 Show that the metric g on M induced by \mathbb{H}^3 is determined by

$$g_{11} = \frac{(\cos \theta + 1)^2}{\sin^2 \theta}, \quad g_{22} = \frac{1}{\sin^2 \theta}, \quad g_{12} = g_{21} = 0$$

(where $\varphi = x_1, \theta = x_2$).

5.2 Show that the Christoffel symbols are given by

$$\Gamma_{11}^2 = \frac{(\cos\theta + 1)^2}{\sin\theta}, \quad \Gamma_{22}^2 = -\frac{\cos\theta}{\sin\theta}, \quad \Gamma_{12}^1 = \Gamma_{21}^1 = -\frac{1}{\sin\theta},$$

and the remaining symbols are zero.

- **5.3** Compute the sectional curvature of M at any point (φ, θ) .
- **5.4** Let $N \subset \mathbb{H}^3$ be parametrized by (ψ, h) via

$$(x, y, z) = (2\cos\psi, 2\sin\psi, h), \quad \psi \in (0, \pi), h > 0.$$

Show that the map $f: M \to N$ defined by

$$f(\varphi, \theta) = \left(\varphi, \frac{2\sin\theta}{1+\cos\theta}\right)$$

is a local isometry.