Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2020	MATH4171-WE01

Title: Riemannian Geometry IV

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. Show your working and explain your reasoning.

Revision:	

ED01/2020

Q1 **1.1** For given $a \in \mathbb{R}$ denote $M_a = \{(x, y, z) \in \mathbb{R}^3 \mid (x^2 - y^2 + z + 1)^2 = a^2\}$.

(i) Find all $a \in \mathbb{R}$ such that M_a is a 2-dimensional smooth manifold.

(ii) For $a = 1$, find two curves $\gamma_i : [0, 1] \to M_1$, $i = 1, 2$ such that $\gamma_1(0) = \gamma_2(0) = (0, 0, 0)$ and $\{\gamma_1'(0), \gamma_2'(0)\}$ is a basis for $T_{(0,0,0)}M_1$.

1.2 Let X, Y be two vector fields on $\mathbb{R}^3$ given by

$$X(x, y, z) = (x + y)\frac{\partial}{\partial x} - yz\frac{\partial}{\partial z}, \qquad Y(x, y, z) = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}.$$

(i) Compute the Lie bracket $[X, Y]$ of X and Y.

(ii) Let $M = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\}$ be a coordinate plane. Show that the restriction of $[X, Y]$ to M is a vector field on M.

Q2 A geodesic $c : [0, \infty) \to M$ in a Riemannian manifold M is called a *ray starting from* $c(0)$ if it minimizes the distance between $c(0)$ and $c(t)$ for any $t \in (0, \infty)$.

2.1 Define the Ricci curvature $Ric(v)$ on a Riemannian manifold. State the Bonnet – Myers theorem.

2.2 Let M be a complete connected Riemannian manifold. Suppose that there exists an $\varepsilon > 0$ such that $Ric(v) > \varepsilon$ for each $v \in \{v \in TM \mid \|v\| = 1\}$. Show that M contains no ray starting from any point $p \in M$.

2.3 Find an example of a complete Riemannian manifold N of positive Ricci curvature containing a ray starting from p for every point $p \in N$.
(You do **not** need to prove that the curvature of N is positive and you do **not** need to prove completeness of N, but you **need** to show an existence of a ray for every point).

Q3 Consider the set F of functions from $\mathbb{C}$ to $\mathbb{C}$ given by

$$F = \{f_{a,b} : \mathbb{C} \to \mathbb{C} \mid f_{a.b}(z) = az + b,\ a, b \in \mathbb{C},\ a \neq 0\}.$$

3.1 Show that all functions belonging to F form a group G with composition as the group operation.

3.2 Define the notion of a Lie group and show that the group G defined in (a) is a Lie group. Find the dimension of G.

3.3 Given a vector $v \in T_eG$ (where $e \in G$ is the neutral element), find the left-invariant vector field X on G such that $X(e) = v$.

3.4 Let g be the left-invariant metric on G which at the neutral element $f(z) = z$ coincides with the Euclidean metric ($g_{ij} = I$). Find the coefficients of g at any point $h = f_{a_0,b_0} \in G$.

Q4 **4.1** Let ∇ be an affine connection on $M = \mathbb{R}^3$ defined by

$$\Gamma_{12}^3 = \Gamma_{23}^1 = \Gamma_{31}^2 = \Gamma_{21}^3 = \Gamma_{32}^1 = \Gamma_{13}^2 = 1,$$

with all the other Christoffel symbols being zero. Show that this connection is torsion-free.

4.2 Show that the connection defined in (**4.1**) does not have the Riemannian property.

4.3 State the theorem of Hopf – Rinow.

4.4 A Riemannian manifold (M, g) is called *homogeneous* if, for any pair $p, q \in M$, there exists an isometry $f : M \to M$ such that $f(p) = q$. Show that any homogeneous manifold must be complete.

Q5 Let $\mathbb{H}^3 = \{(x, y, z) \in \mathbb{R}^3 \,|\, z > 0\}$ be 3-dimensional hyperbolic space (the model in the upper halfspace), with metric $\tilde{g}$ given by

$$\tilde{g}_{ii} = 1/z^2, \quad \tilde{g}_{ij} = 0 \text{ if } i \neq j.$$

Consider $M \subset \mathbb{H}^3$ parametrized by (φ, θ) via

$$(x, y, z) = ((1 + \cos\theta)\cos\varphi, (1 + \cos\theta)\sin\varphi, \sin\theta), \quad \varphi, \theta \in (0, \pi).$$

5.1 Show that the metric g on M induced by $\mathbb{H}^3$ is determined by

$$g_{11} = \frac{(\cos\theta + 1)^2}{\sin^2\theta}, \quad g_{22} = \frac{1}{\sin^2\theta}, \quad g_{12} = g_{21} = 0$$

(where $\varphi = x_1, \theta = x_2$).

5.2 Show that the Christoffel symbols are given by

$$\Gamma_{11}^2 = \frac{(\cos\theta + 1)^2}{\sin\theta}, \quad \Gamma_{22}^2 = -\frac{\cos\theta}{\sin\theta}, \quad \Gamma_{12}^1 = \Gamma_{21}^1 = -\frac{1}{\sin\theta},$$

and the remaining symbols are zero.

5.3 Compute the sectional curvature of M at any point (φ, θ).

5.4 Let $N \subset \mathbb{H}^3$ be parametrized by (ψ, h) via

$$(x, y, z) = (2\cos\psi, 2\sin\psi, h), \quad \psi \in (0, \pi),\ h > 0.$$

Show that the map $f : M \to N$ defined by

$$f(\varphi, \theta) = \left(\varphi, \frac{2\sin\theta}{1 + \cos\theta}\right)$$

is a local isometry.