Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2020	MATH4241-WE01

Title:

Representation Theory IV

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. Show your working and explain your reasoning.

Revision:	

ED01/2020

Q1 Let (ρ, V) be a representation of a finite group G, and let

$$V^G = \{v \in V : \rho(g)v = v \text{ for all } g \in G\}.$$

1.1 Construct (with proof) a G-homomorphism $\pi : V \to V$ such that $\pi(v) \in V^G$ for all $v \in V$ and $\pi(v) = v$ for all $v \in V^G$.

1.2 Using part **1.1**, prove that there is a subrepresentation $W \subset V$ such that $V = V^G \oplus W$. *You do not need Maschke's theorem and may not assume it.*

1.3 Let χ be the character of V and let $\mathbf{1}$ be the trivial character. By taking the trace of the map π from **1.1**, prove that

$$\dim V^G = \langle \chi, \mathbf{1} \rangle .$$

1.4 Suppose further that V is an irreducible and nontrivial representation of G. Prove that

$$\sum_{g \in G} \rho(g) = 0.$$

Q2 Let G be the group of order 20 generated by two elements x and y subject to the relations

$$x^4 = e, \quad y^5 = e, \quad xyx^{-1} = y^2.$$

Its elements are

$$\{x^a y^b \, : \, 0 \le a \le 3, 0 \le b \le 4\}$$

and its conjugacy classes are:

$$\{e\}, \quad \{y, y^2, y^3, y^4\}, \quad \{xy^i : 0 \le i \le 4\}, \quad \{x^2y^i : 0 \le i \le 4\}, \quad \text{and} \quad \{x^3y^i : 0 \le i \le 4\}.$$

2.1 Show that G has a normal subgroup H with $G/H \cong C_4$.

2.2 Hence find the character table of G.

Now let (ρ, V) be the irreducible representation of G of largest dimension.

2.3 By restricting ρ to the subgroup $\langle y \rangle$, or otherwise, show that $\rho(y)$ is diagonalizable and find its eigenvalues.

2.4 Find matrices A and B such that, with respect to some basis of V,

$$\rho(y) = A$$

and

$$\rho(x) = B.$$

Hint: choose a basis so that $\rho(y)$ is diagonal.

Q3 Let $\mathfrak{g}$ be the Lie algebra

$$\left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in \mathbb{C} \right\}.$$

Let $X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, and $Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

3.1 Compute $[X, Y]$. Show that $[A, Z] = 0$ for all $A \in \mathfrak{g}$.

3.2 If (ρ, V) is an irreducible finite-dimensional representation of $\mathfrak{g}$, apply Schur's lemma to show that $\rho(Z)$ is a scalar.

3.3 By considering $\operatorname{tr} \rho(Z)$, show that if (ρ, V) is an irreducible finite-dimensional representation of $\mathfrak{g}$ then $\rho(Z) = 0$.

3.4 Let $V = \mathbb{C}[x]$ be the (infinite-dimensional) vector space of complex polynomials in one variable. There is a representation ρ of $\mathfrak{g}$ on V for which

$$(\rho(X)f)(x) = f'(x)$$

and

$$(\rho(Y)f)(x) = xf(x).$$

What is $\rho(Z)$ for this representation? *Here f' denotes the derivative of f.*

3.5 Prove that V is irreducible. *Hint: Suppose that $W \subset V$ is a nonzero subrepresentation. Show that W must contain a nonzero constant, and then that $W = V$.*

Q4 **4.1** Let $V = \mathbb{C}^2$ be the standard representation of $\mathfrak{sl}_{2,\mathbb{C}}$. Decompose the representation $V \otimes V \otimes V$ into irreducibles.

Now let $\mathfrak{g} = \mathfrak{sl}_{3,\mathbb{C}}$, and let $V = \mathbb{C}^3$ be the standard representation of $\mathfrak{g}$.

4.2 Let $W = \operatorname{Sym}^2(V) \otimes V$ where $V = \mathbb{C}^3$ is the standard representation of $\mathfrak{g}$. Draw a diagram of the weights of W, showing your working.

4.3 Show that, if e_1, e_2, e_3 is the standard basis of V, then $e_1^2 \otimes e_2 - (e_1 e_2) \otimes e_1$ is a highest weight vector in W.

Q5 **5.1** For which partitions μ does the Specht module S^μ occur as a subrepresentation of the representation M^λ of S_6, where $\lambda = (2, 2, 2)$? State clearly any results you use.

5.2 Let λ be the partition $(2, 2)$. Let $s = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$ and $t = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$. Show that the polytabloids e_s and e_t are a basis for S^λ.

5.3 Let $n \geq 4$. For $\lambda = (n-2, 2)$, prove that M^λ is a direct sum of three distinct irreducible representations.

You may assume that, if χ is the permutation character for a group G acting on a set X, then

$$\langle \chi, \chi \rangle = |\{\textit{orbits of } G \textit{ on } X \times X\}|.$$