Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2021	MATH1031-WE01

Title:

Discrete Mathematics

Time (for guidance only):	2 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. To receive credit, your answers must show your working and explain your reasoning.

Revision:	

ED01/2021

Q1 **1.1** (i) Find the generating function for the sequence $a_n = n\binom{n+2}{n}$, $(n \geq 0)$

(ii) Evaluate $\sum_{n=0}^{\infty} n\binom{n+2}{n}3^{-n}$.

1.2 How many solutions $(x_1, x_2, x_3, x_4, x_5, x_6)$ are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 2,$$

where each x_i is an integer and $x_1 \leq 2$, $x_2 \leq -2$, and $x_i \leq -3 + 2i$ for $i = 3, 4, 5, 6$?

Q2 **2.1** Guests at a party arrive in pairs. Each time a new pair of guests arrives, each one of the two new guests shakes hands with each of the guests who are already in attendance (the two new arrivals do not shake hands with each other).

Let h_n be the total number of handshakes that occur if n pairs of guests attend the party, for $n \geq 0$ an integer. So $h_0 = 0$, $h_1 = 0$ (there are no hands to shake for the first pair to arrive), and $h_2 = 4$.

(i) Derive (but do not solve) a recurrence relation for h_n, valid for $n \geq 1$.

(ii) Let $g(x)$ denote the generating function for the sequence h_n: $g(x) = \sum_{n=0}^{\infty} h_n x^n$.
Use your recurrence relation to derive an algebraic equation for $g(x)$, and hence find a closed-form expression for $g(x)$.

(iii) Use your formula for $g(x)$ to find the value of h_{19}.

2.2 Recall that a derangement of a word is an arrangement of the letters in which no letter is in the correct position. Use the Inclusion-Exclusion principle to find the number of derangements of LEMMA.

Q3 **3.1** Solve the recurrence relation

$$a_n = 2a_{n-1} + 3a_{n-2} + 4n - 20, \quad (n \geq 2),$$

with initial conditions $a_0 = 2$ and $a_1 = 7$.

3.2 Let a_n be the number of ways of making n pence from any combination of 5 pence coins and 10 pence coins.

(i) Write down a generating function for a_n and express it as compactly as possible.

(ii) Use your generating function to find a_{200}.

Q4 Answer each of the following questions carefully with proper explanation. You can use theorems and lemmas covered in the lecture notes. Please write neatly.

4.1 Let P_{2n} be the path graph on $2n$ vertices. Let S be a subset of $n + 1$ vertices of P_{2n}. Prove that there are two vertices in S with an edge connecting them.

4.2 Let G be a planar graph with v vertices, e edges and f faces. Prove that $f \leq 2v - 4$.

4.3 An Eulerian tour in a graph G is a walk that traverses every edge exactly once (but it need not be closed). Let G be a connected graph with two vertices s and t having odd degree and all other vertices having even degree. Prove that G has an Eulerian tour.