Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2021	MATH1051-WE01

Title: Analysis I

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. To receive credit, your answers must show your working and explain your reasoning.

Revision:	

ED01/2021

Q1 **1.1** Let $a, b, c, d \in \mathbb{R}$ with $b, d > 0$ and $\frac{a}{b} < \frac{c}{d}$. Show that

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}.$$

1.2 Let $n \geq 2$ be an integer. Show that

$$\left(1+\frac{1}{n-1}\right)^n > \left(1+\frac{1}{n}\right)^{n+1} \text{ is equivalent to } \left(1+\frac{1}{n^2-1}\right)^n > 1+\frac{1}{n}.$$

1.3 Let $n \geq 2$ be an integer. Show that

$$\left(1+\frac{1}{n-1}\right)^n > \left(1+\frac{1}{n}\right)^{n+1}.$$

Q2 **2.1** Let M be a non-empty set of real numbers which is bounded above, and let $a = \sup M$. Given $\varepsilon > 0$, show that there exists an $m \in M$ with

$$0 \leq a - m < \varepsilon.$$

2.2 Show that the set

$$M = \left\{ \frac{m^3 + 12n^2 m}{8n^3 + 6nm^2} \in \mathbb{R} \;\middle|\; n, m \in \mathbb{N}, m \leq 2n \right\}$$

is bounded, and determine its infimum and supremum. Furthermore, decide whether the set has a maximum or minimum.

Q3 **3.1** Let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence of real numbers with limit $x \in \mathbb{R}$. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence of real numbers with

$$y_n - \frac{1}{n} < x_n < y_n + \frac{1}{n}$$

for all $n \in \mathbb{N}$. Show that $(y_n)_{n\in\mathbb{N}}$ converges to x, using the ε-definition of convergence.

3.2 Let $\alpha \in \mathbb{R}$ with $1 < \alpha$. Define $a_1 = 2\alpha$, and for $n \in \mathbb{N}$ let

$$a_{n+1} = \frac{a_n^2 - \alpha}{2a_n - (\alpha+1)}.$$

Show that $(a_n)_{n\in\mathbb{N}}$ is convergent, and determine $\lim_{n\to\infty} a_n$. Hint: First show that $a_n \geq \alpha$ for all $n \in \mathbb{N}$.

Q4 **4.1** Calculate $\liminf_{n\to\infty} x_n$ and $\limsup_{n\to\infty} x_n$ of the sequence $(x_n)_{n\in\mathbb{N}}$ given by

$$x_n = \begin{cases} (-1)^n + 1 & n = 3k \text{ with } k \in \mathbb{N} \\ \left((-1)^n + \dfrac{1}{n}\right)^n & n = 3k - 1 \text{ with } k \in \mathbb{N} \\ \dfrac{n^2 + n - 1}{(2n+1)(n+2)} & n = 3k - 2 \text{ with } k \in \mathbb{N} \end{cases}$$

4.2 Let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence of real numbers, and let $a = \liminf_{n\to\infty} a_n$. Given $\varepsilon > 0$, show that there are at most finitely many $n \in \mathbb{N}$ with $a_n \leq a - \varepsilon$.

Q5 **5.1** Decide whether the series $\sum_{n=1}^{\infty} a_n$ converges, where

$$a_n = \frac{1}{\binom{5n}{4n}}.$$

5.2 Determine all $\alpha \in \mathbb{R}$ for which

$$\sum_{n=1}^{\infty} \frac{\sqrt{(n+1)^3} - \sqrt{n^3}}{n^\alpha}$$

is convergent.

Q6 **6.1** Let f be a continuous function on $\mathbb{R}$ with $f(2) = f(0)$. Consider $f(x) - f(x+1)$ to show that there exists an $x_0 \in [0,1]$ such that $f(x_0 + 1) = f(x_0)$.

6.2 Let f be a continuous function on $[0,\infty)$ with $\lim_{x\to\infty} f(x) = L \in \mathbb{R}$. Show f is bounded. Does the statement hold for f continuous on $(0,\infty)$ with $\lim_{x\to\infty} f(x) = L$?

Q7 **7.1** For $n \in \mathbb{N}$ with $n \geq 2$ define

$$f(x) = \begin{cases} x^n \sin(1/x^2) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Show f is differentiable at $x = 0$. Determine for which n the derivative f' is continuous at $x = 0$.

7.2 Let $a < b$ be two (real) solutions for the equation $e^x \sin x = 1$. Show there exists (at least) one solution of $e^x \cos x = -1$ in the interval (a, b). (Hint: Consider the function $e^{-x} - \sin x$.)

Q8 **8.1** Show that the power series $f(x) = \sum_{k=1}^{\infty} \frac{1}{k} x^k$ converges absolutely for $|x| < 1$ but diverges at $x = 1$.

8.2 Let $\varepsilon > 0$. Show that for $N \in \mathbb{N}$ with $1/N < \varepsilon$ we have

$$0 < (1 - x) \sum_{k=N}^{\infty} \frac{1}{k} x^k < \varepsilon$$

for all $x \in [0, 1)$.

8.3 Use **8.2** to show

$$\lim_{x \to 1^-} (1 - x) f(x) = 0.$$

(Hint: $\delta = 1/N^2$ will do).

Q9 **9.1** For $n \in \mathbb{N}$ let $f_n(x) = \frac{1}{x^{n+1}}$ on $I = [1, \infty)$. Show that f_n converges uniformly on $[r, \infty)$ for all $r > 1$. Is the convergence also uniform on I?

9.2 Let $g(x)$ be a bounded and continuous function on $(0, \infty)$. Let $r \geq 1$. Show that the improper integral

$$\int_r^{\infty} g(x) f_n(x) dx.$$

exists for all $n \in \mathbb{N}$, and

$$\lim_{n \to \infty} n \int_r^{\infty} g(x) f_n(x) dx = 0 \qquad \text{if } r > 1.$$

Give, with proof, two counterexamples (which do not differ by a constant) for the last statement if $r = 1$ (but $g(x)$ still bounded and continuous).

Q10 For $x \in \mathbb{R}$ let $s(x)$ be the distance of x to the 'closest' integer. You may assume that $s(x)$ is 1-periodic and continuous given on $[0, 1]$ by

$$s(x) = \begin{cases} x & \text{if } x \in [0, 1/2] \\ 1 - x & \text{if } x \in [1/2, 1]. \end{cases}$$

10.1 Set

$$f(x) = \sum_{k=0}^{\infty} \frac{s(4^k x)}{4^k}.$$

Show that $f(x)$ defines a continuous function on $\mathbb{R}$.

10.2 Explicitly compute $\int_0^1 f(x) dx$, justifying all steps.

