Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2021	MATH1061-WE01

Title:
Calculus I

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. To receive credit, your answers must show your working and explain your reasoning.

Revision:	

ED01/2021

Q1 Calculate

$$\int_0^{1/2} \frac{(16x^2 - 8x + 1)\exp(\cos(\pi x))}{\exp(\cos(\pi x)) + \exp(\sin(\pi x))}\, dx.$$

Q2 Consider a third order ordinary differential equation for $y(x)$ of the form

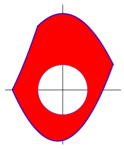
$$y''' - 2y'' - y' + 2y = \phi,$$

where ϕ is some known function of x. Derive integral formulae for the functions u_1, u_2, u_3, such that

$$y = u_1 e^x + u_2 e^{-x} + u_3 e^{2x}$$

solves the above differential equation. In each case the integrand should be a given function of x multiplied by ϕ.

Q3 The region D is shown as the shaded region in the figure. The boundary of D consists of the unit circle centred at the origin, together with parts of the curves $2y = x^2 + 8x + 12$ and $2y = 6 - x^2$ and $8y = 5x^2 + 2x - 16$. Calculate $\iint_D x\, dxdy$.



Q4 The function $g(x)$ has period 2π and its Fourier series is given by

$$\frac{1}{3}\pi^2 + 4\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2} \cos(nx).$$

Without finding $g(x)$, calculate the integral $\int_0^{\pi} (g-3)^2\, dx$. You may use the result that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Q5 Given a function $f(x)$, that is differentiable at least three times for all $x \in [0,1]$, define the associated quadratic polynomial $Y_2(x)$ by the properties that $f(0) = Y_2(0)$ and $f(1) = Y_2(1)$ and $f'(0) = Y_2'(0)$.

Give a formula for $Y_2(x)$ and derive a Lagrange-type formula for the remainder $r_2(x)$, when $Y_2(x)$ is used to approximate $f(x)$ in the interval $(0,1)$.

For the example with $f(x) = e^x$, calculate $Y_2(x)$ and prove that $P_2(x) \leq Y_2(x)$, where $P_2(x)$ is the Taylor polynomial of order two about $x = 0$ for $f(x)$. Furthermore, for this example, use your result for $r_2(x)$ to prove that $f(x) \leq Y_2(x)$ for all $x \in (0,1)$.

Q6 (6.1) Find the equation of the tangent plane T to the surface

$$f(x, y, z) = -xy^2 - z(x + y) + 6y - z^3/3 + 12 = 0$$

at the point $(x, y, z) = (0, -1, 3)$.

(6.2) Find the terms in the Taylor series of the function $f(x, y) = \ln(1+x-y)\sin(xy)$ expanded about the point $(x_0, y_0) = (1, 0)$ to quadratic order.

Q7 (7.1) Find and classify the stationary points of the function

$$f(x, y) = (x^3 - 3x)(y - 1) + \frac{1}{3}y^3 - y.$$

(7.2) A kitchenware supplier manufactures lidless saucepans in the shape of a cylinder of height h and radius r. The material for the bottom of the saucepan costs α/unit area, while the material for the side of the saucepan costs β /unit area. If the total cost of the material in the saucepan is C , use the method of Lagrange multipliers to calculate the maximum possible volume V of the saucepan, in terms of C, α and β.

Q8 Explain briefly why the solutions to the second order linear differential equation

$$(x^2 - 2)\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + \lambda y = 0.$$

can be expressed as a power series $y = \sum_{n=0}^{\infty} a_n x^n$. Find a recurrence relation satisfied by the coefficients a_n and use the ratio test to determine for which values of x the series converges. In the case that $\lambda = 2$ find an explicit expression (i.e. not as an infinite series) for the general solution $y(x)$ in terms of x and the parameters a_0 and a_1.

Q9 A bar of metal lies along the x-axis between $x = 0$ and $x = \pi$. Its temperature $u(x, t)$ satisfies the equation $u_t = k^2 u_{xx}$. Both ends of the bar are insulated so that $u_x(0, t) = u_x(\pi, t)$, and initially the temperature of the bar is $u(x, 0) = 100x$.

Use the method of separation of variables to obtain the solution $u(x, t)$ as a series in the form

$$u(x, t) = \sum_{n=0}^{\infty} A_n X_n(x) T_n(x),$$

where $X_n(x)$ satisfies $X'' = \lambda X$ and $X_n'(0) = X_n'(\pi)$. (You may use that the operator $\mathcal{L} = \frac{d^2}{dx^2}$ is self adjoint with respect to the inner product $(f, g) = \int_0^\pi f(x)g(x)dx$ if $f'(0) = g'(0) = f'(\pi) = g'(\pi) = 0$.)

Q10 The function $y(t)$ satisfies the first order differential equation.

$$\frac{dy}{dt} + \alpha y = f(t)$$

where $\alpha > 0$.

By taking the Fourier transform of this equation, find an expression for $\tilde{y}(p)$, the Fourier transform of $y(t)$, in terms of $\tilde{f}(p)$, the Fourier transform of $f(t)$. Use the convolution theorem to find $y(t)$ in terms of $f(t)$. You may use that the Fourier transform of

$$g(t) = \begin{cases} e^{-t} \text{ for } t > 0 \\ 0 \text{ for } t < 0 \end{cases}$$

is $\tilde{g}(p) = (ip + 1)^{-1}$.

Find the solution $y(t)$ for $t > \beta > 0$ in the case that

$$f(t) = \begin{cases} 1 \text{ for } 0 < t < b \\ 0 \text{ otherwise.} \end{cases}$$