Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2021	MATH1071-WE01

Title: Linear Algebra I

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. To receive credit, your answers must show your working and explain your reasoning.

Revision:	

ED01/2021

Q1 **1.1** Find three linear equations in three variables, i.e., equations of the form $ax+by+cz = d$ with a, b, c, $d \in \mathbb{R}$, such that any two of them have an infinite number of common solutions, but all three have no solution in common. Briefly justify your answer.

1.2 Let L_1 and L_2 be two non-intersecting, non-parallel lines in $\mathbb{R}^3$, neither of which pass through the origin. Consider the set

$$X = \left\{ \mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} = \mathbf{u} + \mathbf{v} \text{ for some } \mathbf{u} \in L_1 \text{ and } \mathbf{v} \in L_2 \right\} .$$

Depending on L_1 and L_2, which of the following may be true? For those you think can happen, give an example; for those you think cannot happen, prove your assertion. Justify your answers carefully.

(i) X is the whole of $\mathbb{R}^3$;
(ii) X is a 2 dimensional vector subspace of $\mathbb{R}^3$;
(iii) X is a 2 dimensional affine subspace of $\mathbb{R}^3$;
(iv) X is a 1 dimensional vector subspace of $\mathbb{R}^3$;
(v) X is a 1 dimensional affine subspace of $\mathbb{R}^3$;
(vi) X is none of the above.

Q2 **2.1** For what values of $s, t \in \mathbb{R}$ is the matrix

$$A = \begin{pmatrix} 2 & 2 & 2 \\ s & 0 & 1 \\ 1 & t & 1 \end{pmatrix}$$

invertible? Justify your answer, including showing any working and stating any results you use from the lecture notes.

2.2 Now let $t = 0$. For those values of s for which you think the matrix is invertible, use the Gauss-Jordan algorithm to find the inverse of A as a function of s. Show your working fully.

Q3 The set A of vectors given below span a subspace V of $\mathbb{R}^4$. Find a basis for the orthogonal complement of V (using the standard inner product on $\mathbb{R}^4$), and find a subset of A which is a basis for V, justifying that it is a basis.

$$A = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \\ -3 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\} .$$

Q4 **4.1** Let $\mathbb{R}[x]_n$ be the vector space of polynomials in x with real coefficients and degree at most n. Define the function $\Delta\colon \mathbb{R}[x]_n \to \mathbb{R}[x]_n$ by

$$\Delta(f(x)) = f(x) - f(x+1) .$$

Prove that Δ is a linear map.

4.2 Find a basis for each of the kernel and the image of Δ, justifying your answer. Now describe the image of Δ^m (the composite of Δ with itself, m times) for each positive integer m, explaining your reasoning.

Q5 **5.1** Let $\{e_1, e_2, e_3, e_4\}$ be the standard basis for $\mathbb{R}^4$, that is, $e_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$, $e_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$, etc. Let $x_1 = \begin{pmatrix}1\\1\\0\\1\end{pmatrix}$ and $x_2 = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}$. Use the process applied in the proof of the Steinitz Exchange Theorem (Theorem 6.2.9 of the Michaelmas lecture notes) to find a basis for $\mathbb{R}^4$ containing x_1, x_2 and some of the e_i. Do this three times to obtain three distinct bases. Show your working.

5.2 Suppose the finite dimensional real vector space V is the direct sum $U_1 \oplus U_2$ of subspaces U_1 and U_2. Suppose given linear maps $f_1 \colon U_1 \to W$ and $f_2 \colon U_2 \to W$. Form the linear map $F \colon V \to W$ by

$$F(u_1 + u_2) = f_1(u_1) + f_2(u_2) \quad \text{for} \quad u_1 \in U_1,\ u_2 \in U_2\,.$$

If f_i has rank r_i, express the rank of F in terms of r_1, r_2 and any other number or numbers you consider relevant, and state what the full range of possible values this rank may take. Briefly explain your reasoning, including stating any results you use from lectures.

Q6 **6.1** Find all the values of $r \in \mathbb{R}$ for which the matrix

$$A_r = \begin{pmatrix}0 & 1\\0 & r\end{pmatrix}$$

is diagonalizable.

6.2 For the same matrices A_r as in the previous question, find all the values of $r \in \mathbb{R}$ such that A_r is orthogonally diagonalizable.

6.3 Give an example for every integer $n > 2$ (justifying your answer) of a non-diagonalizable $n \times n$ matrix B_n such that

$$B_n^2 \prod_{k=3}^{n} (B_n - kI) = B_n^2(B_n - 3I)(B_n - 4I) \cdots (B_n - nI) = 0\,.$$

Q7 Consider the subspace U of $\mathbb{R}^3$ (with its standard inner product) given by vectors (x, y, z) satisfying

$$2x - y + z = 0\,.$$

7.1 Find an orthonormal basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ for U, with $\boldsymbol{u}_1$ proportional to $\begin{pmatrix}1\\2\\0\end{pmatrix}$.

7.2 Find a third vector $\boldsymbol{u}_3$ so that $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is an orthonormal basis for $\mathbb{R}^3$.

Q8 Consider the vector space Sym_n of real symmetric $n \times n$ matrices.

8.1 Show that

$$(u, v) = \sum_{i=1}^{n} \sum_{j=1}^{n} u_{ij} v_{ji}$$

for $u, v \in \mathrm{Sym}_n$ defines an inner product on Sym_n.

8.2 Consider the case of real symmetric 2×2 matrices Sym_2 (with the inner product as above), and define D_2 to be the subspace of diagonal matrices. Find the matrix in D_2 nearest to

$$w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

Q9 Consider the vector space $\mathbb{R}[x]$ with inner product

$$(f, g) = \int_{-1}^{1} (1 - x^2) f(x) g(x) dx$$

and the family of differential operators $\mathcal{L}_k : \mathbb{R}[x] \to \mathbb{R}[x]$

$$\mathcal{L}_k = (1 - x^2)\frac{d^2}{dx^2} - kx\frac{d}{dx}$$

with $k \in \mathbb{R}$.

9.1 Show that $\mathcal{L}_k$ is a symmetric operator if and only if $(\mathcal{L}_k(x^2), 1) = (x^2, \mathcal{L}_k(1))$. Find the set $S = \{k \text{ such that } \mathcal{L}_k \text{ is symmetric}\}$.

9.2 Restricting yourself to the space of quadratic polynomials $\mathbb{R}[x]_2$, compute the matrix representation, eigenvalues and eigenvectors of $\mathcal{L}_\sigma$ for all $\sigma \in S$.

Q10 Define the set of matrices

$$G = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \text{ with } a \in \mathbb{R} \right\} .$$

Determine whether G together with the following group operations is a group.

(i) Matrix multiplication.

(ii) Matrix addition.

(iii) The product $H(A, B)$ defined for $A = (a_{ij})$, $B = (b_{ij})$ by $H(A, B)_{ij} = a_{ij} b_{ij}$.