Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2021	MATH1551-WE01

Title:

Maths For Engineers and Scientists

Time (for guidance only):	3 hours	
Additional Material provided:	Formula sheet (two pages)	
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. To receive credit, your answers must show your working and explain your reasoning.

Revision:	

ED01/2021

Q1 Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & -3 & 0 \\ -7 & 0 & 0 & -3 \\ -2 & 0 & 0 & -1 \\ 0 & -2 & 7 & 0 \end{bmatrix}$$

1.1 Show that A is invertible.

1.2 Show that A does not have an LU decomposition.

1.3 Find a matrix P such that PA does have an LU decomposition. (There is no need to find L and U, but you should justify your answer.)

1.4 Find the eigenvalues of A. What is their relation to the determinant?

Q2 **2.1** Find all complex numbers z that satisfy each of the following equations.

(i) $z^2 = z^{1/2}$.

(ii) $\sin(z) = \cos(z)$.

(iii) $\sin(z) = i\cos(z)$.

2.2 A square in the complex plane has centre at $-3 - 2i$ and a vertex at $1 + i$.

(i) Find the other vertices.

(ii) Calculate the area of the square.

Q3 **3.1** Show that $\sin(5\theta) = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$, and find a similar expression for $\cos(5\theta)$ in terms of $\cos\theta$.

3.2 Use **3.1** to find an <u>exact</u> expression for $\tan\frac{\pi}{5}$.

3.3 Hence show that the area of a regular pentagon of side length a is $\dfrac{5a^2}{4\sqrt{5-2\sqrt{5}}}$.

Q4 Three planes are given by the vector equations

$$(\boldsymbol{x} - \boldsymbol{a}) \cdot \boldsymbol{n} = 0,$$
$$(\boldsymbol{x} \times \boldsymbol{a}) \cdot \boldsymbol{n} = 0,$$
$$\boldsymbol{x} = \boldsymbol{a} + \lambda\boldsymbol{n} + \mu\boldsymbol{a} \times \boldsymbol{n},$$

where $\boldsymbol{a} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{n} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$.

4.1 Find the Cartesian equation of each plane.

4.2 Determine whether or not the three planes intersect at a single point, and if so find it.

Q5 **5.1** Calculate $\lim_{x \to 0} \frac{\sin^2(3x)}{x\sin(2x)}$ stating any standard result you use.

5.2 Using Calculus of Limits (and without using Taylor's theorem or l'Hôpital's rule), show that

$$\lim_{t \to 0} \frac{\sqrt{1+t}-1}{t} = \frac{1}{2}.$$

Deduce that the function $f(x) = \sqrt{x}$ is differentiable for all $x > 0$ and find its derivative from the first principles.

5.3 State the rule for finding the nth derivative of a product $f(x) = u(x)v(x)$ and use it to find the third derivative of $f(x) = e^x \cosh(x)$ with respect to x. Simplify your answer.

Q6 **6.1** Find integer n so that the degree n Taylor polynomial $P_{n,0}(x)$ approximates $f(x) = e^x$ up to two decimal places for each x in $[-0.5, 0.5]$. Use $P_{n,0}(x)$ to approximate $e^{0.25}$. You may use wihout proof that $e^{0.5} < 2$.

6.2 Fix $A > 0$. Argue that the equation $xe^x = A$ has a unique positive solution. Starting with the initial approximation $x_0 = A$, use the Newton-Raphson method to find the first improvement x_1.

Q7 **7.1** If $f(x, y)$ is a function of x and y, and if $x = e^{-v}\sin(3u)$ and $y = e^{-v}\cos(3u)$, use the Chain Rule to express $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ in terms of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

7.2 Let $r(x)$ be the absolute value of the complex number $z = 5 + ix$. Carefully derive the Taylor series about $x = 0$ for the function $r(x)$. For which values of x does your series converge?

7.3 Find all critical points of the function

$$f(x, y) = (x-1)^2 + \cos^2(y).$$

Determine which of these points are local maxima, local minima, or saddle points.

Q8 **8.1** Find the general solution of

$$y'' - 4y' + 4y = -2 + 12x - 4x^2$$

and hence find the solution satisfying $y(0) = 2$, $y'(0) = 6$.

8.2 For a function $f(r, \theta)$, consider the equation

$$r^2 \frac{\partial^2 f}{\partial r^2} + r\frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial \theta^2} = 0. \qquad (1)$$

Show that the function $f(r, \theta) = r\sin\theta$ solves (1), but neither $f_1(r, \theta) = r$ nor $f_2(r, \theta) = \sin\theta$ do.

8.3 Given integer $n \geq 0$ find real α so that $f(r, \theta) = r^\alpha \cos(n\theta)$ solves (1) and satisfies $f(0, \theta) = 0$ for all θ.

8.4 Given integer $n \geq 0$ find the most general solution of (1) of the form

$$f(r, \theta) = Ar^\alpha \cos(n\theta) + Br^\beta \sin(n\theta) \quad \text{with} \quad f(0, \theta) = 0,$$

where A, B, α, and β are real numbers.