EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2021	MATH1561-WE01

Title:
Single Mathematics A

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. To receive credit, your answers must show your working and explain your reasoning.

Revision:	

ED01/2021

We re-emphasise here that in all questions credit will be given for the derivation of the result and the quality of your explanations. Just giving the answer with no explanation or detail of working will give little credit.

Q1 **1.1** Compute the real part, imaginary part, modulus and argument of

$$-\frac{1}{1-i}, \qquad -\frac{1}{1-\frac{1}{1-i}}, \qquad -\frac{1}{1-\frac{1}{1-\frac{1}{1-i}}}\ .$$

1.2 Find the general solution of

$$e^z = -\frac{1}{1-i}\ .$$

1.3 For which values of a is $z = 1$ a solution of the equation

$$a^2z^4 + 3a^2 - az - 3a - z^5 + 1 = 0?$$

For each of these values of a find all other solutions z of this equation written in the polar repesentation.

Q2 **2.1** Find the values of b for which the following has a finite limit and obtain the limit in each case

$$\lim_{x\to 3} \frac{x^2-(2b-1)x+b(b-1)}{2x^2-5x-3}\ .$$

You may use any method but must explain carefully what you are doing.

2.2 Find

$$\lim_{x\to\pi} \left(1+\frac{\ln(-\cos x)}{\ln(\sin\frac{x}{2})}\right)^2\ .$$

You can use any method but must explain what you are doing.

2.3 Find

$$\frac{d}{dx}\left(\frac{1}{\arccos(x)}\right)\ .$$

You may use, without derivation, the basic rules of differentiation, eg the chain rule and $d/dx(\cos(x)) = -\sin(x)$, as well as basic trig identities.

Q3 **3.1** Find the area of the finite region enclosed by the two curves $y = x^2$ and $y = 1 + \cos^4(\pi x/2)$.

3.2 Find

$$\int \frac{x^4+1}{x^3+x^2+2x+2}dx\ .$$

Q4 **4.1** Determine whether or not the series

$$\sum_{k=1}^{\infty} k!\, e^{-k^2}, \qquad \sum_{k=1}^{\infty} \frac{1}{k\sin^2 k}$$

converge.

4.2 Determine the interval of convergence for the power series

$$\sum_{k=1}^{\infty} k^n x^k,$$

where n is a real number. Find the values of n for which the series converges at the endpoints of the interval of convergence.

Q5 5.1 For the function $f(x) = \sqrt{\cos(x)}$,

(i) Show that the second-order Taylor polynomial about $x = 0$ is $p_2(x) = 1 - \frac{1}{4}x^2$.

(ii) Give a bound on the error $f(x) - p_2(x)$ for $x \in (0, \frac{\pi}{4})$.

(iii) What happens if we try to extend the range to $x \in (0, \frac{\pi}{2})$?

5.2 Let N be a square matrix satisfying $N^3 = 0$ and let k be a non-negative integer. Find integers s and t depending on k such that

$$(I + N)^k = I + sN + tN^2.$$

Use your result to find the explicit form of the kth power

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}^k .$$

Q6 For which values of $r \in \mathbb{R}$ does the system of linear equations

$$\begin{cases} rx - y + rz = r \\ x - ry + rz = r \\ x - y + r^2 z = r \end{cases}$$

have (a) no solutions, (b) infinitely many solutions, (c) a unique solution?

Find all the solutions in the cases (b) and (c).

Q7 7.1 Find complex numbers α, β, γ and δ such that the matrix

$$A = \begin{pmatrix} 0 & \alpha & \beta \\ 1 & \gamma & \delta \\ -1 & i & 0 \end{pmatrix}$$

is Anti-Hermitian. For these values, use Gaussian elimination to construct the inverse of A or explain why the inverse doesn't exist.

7.2 Given $\theta \in \mathbb{R}$ and

$$B = \begin{pmatrix} \cos 2\theta & 0 & \sin 2\theta \\ 0 & -1 & 0 \\ \sin 2\theta & 0 & -\cos 2\theta \end{pmatrix},$$

find an orthogonal matrix P such that $P^T BP$ is diagonal. (You may use the trigonometric identities $\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$.)

Find $\exp\left(\frac{\pi}{2} iB\right)$, expressing your answer in as simple a form as possible.