Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2021	MATH1571-WE01

Title:
Single Mathematics B

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. To receive credit, your answers must show your working and explain your reasoning.

Revision:	

ED01/2021

Q1 A positively charged particle P of mass m and charge $e > 0$ is subject to both an electric field $\boldsymbol{E}$ and a magnetic field $\boldsymbol{B}$. The velocity and acceleration of the particle P throughout its trajectory are given in spherical coordinates by

$$\boldsymbol{v} = \frac{\sqrt{3}e}{m}\omega\,\boldsymbol{e}_\phi \qquad \text{and} \qquad \mathbf{a} = -\frac{3e}{m}\omega^2\,\boldsymbol{e}_r - \frac{\sqrt{3}e}{m}\omega^2\,\boldsymbol{e}_\theta,$$

where $\omega > 0$ is a real constant. Assume that the magnetic field $\boldsymbol{B}$ is given by $\boldsymbol{B} = -B\,\boldsymbol{e}_r$ for some real constant $B > 0$.

1.1 Show that the electric field $\boldsymbol{E}$ is never orthogonal to the magnetic field $\boldsymbol{B}$.

1.2 For which value (or values) of $B > 0$ is the electric field $\boldsymbol{E}$ parallel to the magnetic field $\boldsymbol{B}$? Give an expression for the electric field $\boldsymbol{E}$ in this case.

1.3 Find the angle α between the magnetic field $\boldsymbol{B}$ and the total force vector acting on the particle P.

1.4 The spherical coordinates of the position of the particle P at initial time $t = 0$ are given by $(r, \theta, \phi) = (e/m,\ \pi/3,\ 0)$. Show that the particle P travels on a circular trajectory and find the rate of change of the ϕ-coordinate throughout this trajectory.

Q2 **2.1** The functions $x(t)$ and $y(t)$ satisfy the following equations.

$$\dot{x} - x + 2y = 2e^{-5t}$$
$$\dot{y} - 4x + 5y = -30.$$

Find $x(t)$ and $y(t)$ if $x(0) = 15$ and $y(0) = 8$.

2.2 The population $P(t)$ of king lizards on a remote island evolves according to the differential equation

$$\frac{dP}{dt} = \alpha P - \beta P^3.$$

where α and β are positive constants. Find the general solution to this equation. If $P(0) = 500$, $P(1) = 1000$ and $P(\infty) = 2000$, determine the constants α and β.

Q3 **3.1** Freddie owns a car whose shock absorbers have failed. He travels along a road which is initially level at constant horizontal speed v. The car's vertical displacement from equilibrium $z(t)$ obeys the equation $\ddot{z}+4z = 0$. The 'energy' of the car's vertical motion E is given as

$$E = \frac{1}{2}\left(\dot{z}^2 + 4z^2\right).$$

Show that E is a constant.

3.2 Whilst travelling along the road the car meets with a bump at $t = 0$, at which time the car is vertically at rest so that $z(0) = \dot{z}(0) = 0$. The effect of the bump is to change the equation that $z(t)$ obeys to

$$\ddot{z} + 4z = H\sin\left(\frac{\pi v t}{L}\right)$$

for $0 \leq t \leq L/v$, where H, L are the height and length of the bump and v is the car's horizontal velocity. If after hitting the bump the height of the car $z(t)$ reverts to obeying the equation $\ddot{z} + 4z = 0$, calculate the energy of the vertical motion for $t > L/v$. You may assume that $v \neq 2L/\pi$.

3.3 For what speeds v should Freddie drive at in order to minimise the energy of the vertical motion after hitting the bump? (Again you may assume that $v \neq 2L/\pi$.)

Q4 **4.1** Let $f(x) = 1 - x$ for $0 \leq x < \pi$ and let $f_{even}(x)$ be the even periodic extension of $f(x)$ with period 2π. Sketch the function $f_{even}(x)$ for $-3\pi < x < 3\pi$, and find the Fourier coefficients in the Fourier expansion

$$f_{even}(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx).$$

4.2 The wave equation for a damped wave $u(x,t)$ in an attenuating medium is given by

$$u_{xx} + 2\beta u_x = u_{tt} + 2\alpha u_t,$$

where α and β are positive constants. By writing u in the separated form as a product of functions of x and t,

$$u(x,t) = X(x)T(t),$$

show that the PDE admits travelling wave solutions that are functions of $x \pm ct$ where c is the speed of the wave which you should determine in terms of the wavemumber ($k = 2\pi/\lambda$ where λ is the wavelength) and α and β. Describe what happens to the amplitude of these waves over time and as they propagate through the medium. Determine the maximum value of α above which waves can no longer propagate, and find the value of α for which the waves all travel at the same speed regardless of wavelength.

Q5 A lens is given by the volume enclosed between the paraboloid $z = 1 - x^2/3 - y^2/3$ and the plane $z = 0$. It has a density given by

$$\rho(x,y,z) = x^2 + y^2.$$

Determine the distance between its centre-of-mass and its centroid.

Q6 **6.1** Determine if the following two complex functions are analytic or not:

(i) $f(x,y) = x^3 - 3xy^2 + i(y^3 - 3x^2y)$;

(ii) $g(x,y) = y^3 - 3x^2y + 2xy + 2x^2 - 2y^2 + i(x^3 - 3xy^2 + 4xy - x^2 + y^2)$.

6.2 A ball is released at the point $(x,y) = (1,1)$ on a surface of height

$$f(x,y) = x^4 + 2y^4 - 3x^2.$$

(i) In what direction does it start rolling and what is the slope in that direction?

(ii) Assuming it rapidly loses energy through friction where does it come to rest, and what is the height there?

(iii) The conventional test for a minimum fails at this point: why?

Q7 Jane, a minecraft speedrunner, is planning her schedule for the next few months. Here, we consider a simplified version of the game, where the speedrunner's strategy depends solely on how fast she can convert gold ingots into ender pearls by bartering with piglins. In the simplified game, each gold ingot, when bartered with a piglin, has a chance of 4% of being converted into an ender pearl. To end the game, she needs 14 ender pearls.

7.1 Let X_n denote the random number of ender pearls Jane has after bartering n gold ingots. Name the distribution of X_n, and identify its parameters.

7.2 Calculate the expectation and standard deviation of the number of ender pearls Jane will have after bartering 200 gold ingots. State clearly any formulas that you use from the notes.

7.3 How many gold ingots does she need to ensure that she has a chance of at least 1% of getting at least 14 ender pearls? You may use the normal approximation. State clearly any formulas that you use from the notes.

7.4 Let k^* denote the answer to the previous question. Jane decides always to collect exactly k^* gold ingots on her runs. Find the expectation of the number of these runs she needs until she can finish the game. You may find the following formula useful (you do not need to prove it):

$$\sum_{j=1}^{\infty} j(1-q)^j = \frac{1-q}{q^2}$$

which holds for any number q such that $0 < q < 1$.

7.5 Joe, a competing speedrunner, has beaten Jane's world record and published a speedrun where he collected 150 gold ingots, which he converted into 14 ender pearls. Do you think Joe was cheating? Provide a clear probabilistic argument behind your answer.