Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2021	MATH1597-WE01

Title:
Probability I

Time (for guidance only):	2 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. To receive credit, your answers must show your working and explain your reasoning.

Revision:	

ED01/2021

Q1 A cube has each of its 6 faces inscribed with a random number chosen independently and uniformly at random from the integers $1, 2, 3, 4, 5, 6$.

1.1 What is the probability that each possible number appears exactly once?

1.2 What is the chance that the numbers on each pair of opposite faces sum to 7?

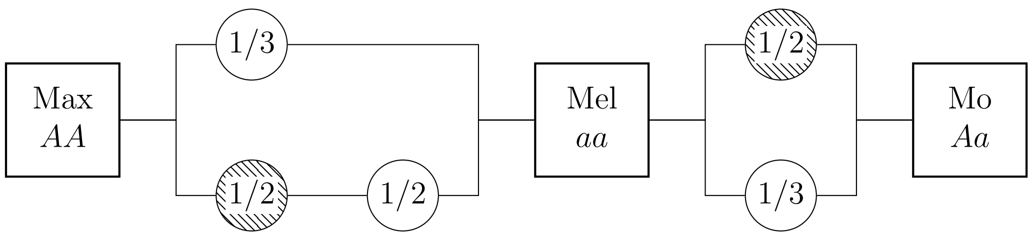
1.3 What is the expected number of pairs of adjacent faces (i.e., faces that share an edge) whose numbers differ by 1?

The cube is now rolled like a standard die, with the upward-facing number being the score X. Let T denote the sum of the numbers on all the faces of the die, and let Y denote the number on the face opposite to the face with the score X (i.e., Y is the downward-facing number).

1.4 What is $\mathbb{E}(X \mid T)$? What is $\mathbb{E}(X)$?

1.5 Are X and Y independent? Are X and Y conditionally independent, given that $T = 6$? How about if $T = 7$? Explain, briefly.

Q2 Two mice, Maximilian and Mortimer, are separated from their potential mate, Melanie, via a maze complex in which doors can be open or closed. The maze layouts, and the probabilities that each door is open, are indicated in the diagram below. Doors are open independently, apart from the two doors shaded in the picture, which are linked and are either both open (probability 1/2), or both closed (probability 1/2).



The gene for fur colour has alleles A and a, with genotypes AA, Aa giving brown fur, and aa giving white fur. Melanie is aa, Maximillian is AA, and Mortimer is Aa. If Maximillian can find a path to Melanie, then Maximillian and Melanie will produce an offspring. If Mortimer can find a path to Melanie, and Maximillian can not, then Mortimer and Melanie will produce an offspring (Maximillian will chase off Mortimer if they both have a path).

2.1 What is the probability that Maximillian and Melanie mate?

2.2 What is the probability that Mortimer and Melanie mate?

2.3 Given an offspring is produced, what is the probability that it has white fur?

2.4 Given that a white offspring is produced, what is the probability that the shaded doors in the maze were open?

2.5 Given that a brown offspring is produced, what is the probability that the shaded doors in the maze were open?

Q3 A bag contains four tokens, labelled from 1 up to 4. Tokens are drawn randomly, one-by-one and without replacement, from the bag, until the 1 token appears. Let X denote the number of tokens drawn, and let Y be the value of the highest-value token seen, both up to and including the appearance of the 1. For example, the sequence of draws $2, 4, 1$ would give $X = 3$ and $Y = 4$.

3.1 Find the joint probability mass function of X and Y.

3.2 Calculate $\mathbb{E}(X)$, $\mathbb{E}(Y)$, and $\mathbb{C}\mathrm{ov}(X, Y)$.

3.3 Find the conditional expectations $\mathbb{E}(Y \mid X = x)$ for $x = 1, 2, 3, 4$.

3.4 Verify that, in this example, $\mathbb{E}(\mathbb{E}(Y \mid X)) = \mathbb{E}(Y)$.

Q4 **4.1** Let $Y \sim \mathrm{Exp}(\beta)$, $\beta > 0$. Compute the moment generating function $M_Y(t) = \mathbb{E}(\mathrm{e}^{tY})$ for all values of $t \in \mathbb{R}$.

4.2 Let $Z \sim \mathcal{N}(0, 1)$ and set $X = Z^2$. Compute the moment generating function $M_X(t) = \mathbb{E}(\mathrm{e}^{tX})$ for all values of $t \in \mathbb{R}$.

4.3 Find $\mathbb{E}(X)$ and $\mathbb{E}(X^2)$.

4.4 Let $S = Z_1^2 + Z_2^2$, where Z_1, Z_2 are independent $\mathcal{N}(0, 1)$ random variables. Compute the moment generating function $M_S(t) = \mathbb{E}(\mathrm{e}^{tS})$ for all values of $t \in \mathbb{R}$.

4.5 Identify the distribution of S.

Q5 Let d be a positive integer and let $U_1, \ldots, U_d$ be independent random variables, uniform on $[0, 1]$. Consider the random (rectangular) cuboid C_d whose side lengths are $U_1, \ldots, U_d$, i.e.,

$$C_d = [0, U_1] \times \cdots \times [0, U_d].$$

Let V_d denote the volume of C_d.

5.1 Calculate the cumulative distribution function of $-\log U_i$. What is the corresponding distribution?

5.2 Show that there is a constant $\lambda > 0$, whose value you should determine, such that, for every $\varepsilon > 0$,

$$\lim_{d\to\infty} \mathbb{P}\left(\left|\frac{\log V_d}{d} + \lambda\right| > \varepsilon\right) = 0.$$

5.3 For the constant λ determined above, find, as a function of $a \in \mathbb{R}$,

$$\lim_{d\to\infty} \mathbb{P}\left(V_d \geq \mathrm{e}^{-\lambda d + a\sqrt{d}}\right).$$

You may express your answer in terms of Φ, the cumulative distribution function of the standard normal distribution.