

EXAMINATION PAPER

Examination Session: May/June

2021

Year:

Exam Code:

MATH2011-WE01

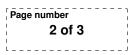
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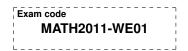
Complex Analysis II

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks.				
	Please start each question on a new	rt each question on a new page.			
	Please write your CIS username at the top of each page.				
	To receive credit, your answers must show your working and explain your reasoning.				
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Revision:





Q1 1.1 Let $a, b, c \in \mathbb{C}$ with $a^2 + b^2 \neq 0$. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is a holomorphic function with

 $\forall z \in \mathbb{C} : a \operatorname{Re}(f(z)) + b \operatorname{Im}(f(z)) + c = 0.$

Show that *f* is constant.

1.2 Find a holomorphic function f = u + iv: $\{z \in \mathbb{C} : \operatorname{Re}(z) > 0\} \rightarrow \mathbb{C}$ with real part

$$u(x, y) = \arctan\left(\frac{y+1}{x}\right), \quad z = x + iy,$$

where we take the branch arctan : $\mathbb{R} \to (-\pi/2, \pi/2)$.

- **1.3** Determine how many solutions the equation $z^4 + 4 3e^{-z} = 0$ has in the first quadrant $\{z \in \mathbb{C} : 0 < \operatorname{Arg}(z) < \pi/2\}$.
- **Q2 2.1** We equip \mathbb{C} with the following metric (you do not need to verify that it is a metric):

$$d(z,w) := \begin{cases} \frac{1}{2}|z-w|, & |z-w| \le 2, \\ 1, & |z-w| > 2. \end{cases}$$
(*)

Find the open ball $B_1(0) = \{z \in \mathbb{C} : d(z, 0) < 1\}$ and its closure in (\mathbb{C}, d) . Is the closure equal to $\{z \in \mathbb{C} : d(z, 0) \le 1\}$? Justify your answer.

- **2.2** Let $\{z_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ and $z \in \mathbb{C}$. Prove that $z_n \to z$ in (\mathbb{C}, d) with respect to the metric d in (*) is equivalent to $z_n \to z$ in $(\mathbb{C}, |\cdot|)$ with respect to the metric induced by $|\cdot|$.
- **2.3** Suppose that t > 0. Prove that $f_n : \{z \in \mathbb{C} : \operatorname{Re}(z) \ge t\} \to \mathbb{C}$ defined by

$$f_n(z) := \tanh(nz) = \frac{\sinh(nz)}{\cosh(nz)}$$

is uniformly convergent to 1 as $n \to \infty$. Is the convergence uniform in $\{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$? Justify your answer.

Q3 3.1 Prove that every Möbius transformation preserving the open right half-plane $\mathbb{H}_R := \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$ is of the form M_T with

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, d \in \mathbb{R}, \quad b, c \in i\mathbb{R}$$

such that det T = 1.

- **3.2** Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Find a Möbius transformation $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ that preserves \mathbb{H}_R and satisfies f(0) = i, f(2) = 2. What is $f(\infty)$? Where in \mathbb{C} is f conformal?
- **3.3** Let *p* be a polynomial with complex coefficients. Using a parametrisation of the unit circle, show that

$$\overline{\rho'(0)}=\frac{1}{2\pi i}\int_{|z|=1}\overline{\rho(z)}\,dz.$$

Find $\int_{|z|=1} \operatorname{Re}(p(z)) dz$.

Q4 4.1 Consider the sequence of entire functions f_n given by

$$f_n(z) = \sin\left(z + \frac{1}{n}\right), \qquad (n \in \mathbb{N}).$$

Let γ denote the simple closed contour tracing the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ anticlockwise, and suppose that for each $k \ge 1$ there is an entire function g_k such that for $n \in \mathbb{N}$

$$g_k\left(\frac{1}{n}\right) = \frac{1}{2\pi i} \int_{\gamma} \left(f_n(z)\right)^k h(z) \, dz,$$

where $h(z) = 1 + 2z^{-1} + z^{-2}$. By evaluating the integral, identify the function g_k for each $k \ge 1$. Justify your answer.

- **4.2** Let *f* be an entire function and let $\omega \in \mathbb{C}$ be some complex number.
 - (i) Show that the function $\frac{f(z) f(\omega)}{z \omega}$ defined on $\mathbb{C} \{\omega\}$ can be analytically extended to an entire function.
 - (ii) Assume additionally that $\left|\frac{f(z)}{z-\omega}\right|$ tends to 0 as |z| tends to infinity. Using part (i) and Liouville's Theorem, identify the function *f* in the case that $f(\omega) = 2021$. Justify your answer.
- **Q5 5.1** Let a, b > 0. By integrating a suitable complex function around a suitable contour, evaluate the integral

$$\int_0^\infty \frac{\sin(ax)}{x(1+b^2x^2)} dx.$$

5.2 Using Rouché's theorem, prove via contradiction that there does not exist a sequence of polynomials that converges uniformly to $h(z) = \frac{1}{z}$ on the unit circle $\{z \in \mathbb{C} : |z| = 1\}$.