Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2021	MATH2011-WE01

Title:
Complex Analysis II

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. To receive credit, your answers must show your working and explain your reasoning.

Revision:	

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Q1 **1.1** Let $a, b, c \in \mathbb{C}$ with $a^2 + b^2 \neq 0$. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is a holomorphic function with

$$\forall z \in \mathbb{C} : \quad a\,\mathrm{Re}(f(z)) + b\,\mathrm{Im}(f(z)) + c = 0.$$

Show that f is constant.

1.2 Find a holomorphic function $f = u + iv : \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\} \to \mathbb{C}$ with real part

$$u(x, y) = \arctan\left(\frac{y+1}{x}\right), \quad z = x + iy,$$

where we take the branch $\arctan : \mathbb{R} \to (-\pi/2, \pi/2)$.

1.3 Determine how many solutions the equation $z^4 + 4 - 3e^{-z} = 0$ has in the first quadrant $\{z \in \mathbb{C} : 0 < \mathrm{Arg}(z) < \pi/2\}$.

Q2 **2.1** We equip $\mathbb{C}$ with the following metric (you do not need to verify that it is a metric):

$$d(z, w) := \begin{cases} \frac{1}{2}|z - w|, & |z - w| \leq 2, \\ 1, & |z - w| > 2. \end{cases} \qquad (*)$$

Find the open ball $B_1(0) = \{z \in \mathbb{C} : d(z, 0) < 1\}$ and its closure in $(\mathbb{C}, d)$. Is the closure equal to $\{z \in \mathbb{C} : d(z, 0) \leq 1\}$? Justify your answer.

2.2 Let $\{z_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ and $z \in \mathbb{C}$. Prove that $z_n \to z$ in $(\mathbb{C}, d)$ with respect to the metric d in $(*)$ is equivalent to $z_n \to z$ in $(\mathbb{C}, |\cdot|)$ with respect to the metric induced by $|\cdot|$.

2.3 Suppose that $t > 0$. Prove that $f_n : \{z \in \mathbb{C} : \mathrm{Re}(z) \geq t\} \to \mathbb{C}$ defined by

$$f_n(z) := \tanh(nz) = \frac{\sinh(nz)}{\cosh(nz)}$$

is uniformly convergent to 1 as $n \to \infty$. Is the convergence uniform in $\{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$? Justify your answer.

Q3 **3.1** Prove that every Möbius transformation preserving the open right half-plane $\mathbb{H}_R := \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ is of the form M_T with

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, d \in \mathbb{R}, \quad b, c \in i\mathbb{R}$$

such that $\det T = 1$.

3.2 Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Find a Möbius transformation $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ that preserves $\mathbb{H}_R$ and satisfies $f(0) = i$, $f(2) = 2$. What is $f(\infty)$? Where in $\mathbb{C}$ is f conformal?

3.3 Let p be a polynomial with complex coefficients. Using a parametrisation of the unit circle, show that

$$\overline{p'(0)} = \frac{1}{2\pi i} \int_{|z|=1} \overline{p(z)}\, dz.$$

Find $\int_{|z|=1} \mathrm{Re}(p(z))\, dz$.

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Q4 **4.1** Consider the sequence of entire functions f_n given by

$$f_n(z) = \sin\left(z + \frac{1}{n}\right), \qquad (n \in \mathbb{N}).$$

Let γ denote the simple closed contour tracing the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ anticlockwise, and suppose that for each $k \geq 1$ there is an entire function g_k such that for $n \in \mathbb{N}$

$$g_k\left(\frac{1}{n}\right) = \frac{1}{2\pi i}\int_\gamma (f_n(z))^k \ h(z)\, dz,$$

where $h(z) = 1 + 2z^{-1} + z^{-2}$. By evaluating the integral, identify the function g_k for each $k \geq 1$. Justify your answer.

4.2 Let f be an entire function and let $\omega \in \mathbb{C}$ be some complex number.

(i) Show that the function $\dfrac{f(z) - f(\omega)}{z - \omega}$ defined on $\mathbb{C} - \{\omega\}$ can be analytically extended to an entire function.

(ii) Assume additionally that $\left|\dfrac{f(z)}{z - \omega}\right|$ tends to 0 as $|z|$ tends to infinity. Using part (i) and Liouville's Theorem, identify the function f in the case that $f(\omega) = 2021$. Justify your answer.

Q5 **5.1** Let $a, b > 0$. By integrating a suitable complex function around a suitable contour, evaluate the integral

$$\int_0^\infty \frac{\sin(ax)}{x(1 + b^2x^2)}dx.$$

5.2 Using Rouché's theorem, prove via contradiction that there does not exist a sequence of polynomials that converges uniformly to $h(z) = \dfrac{1}{z}$ on the unit circle $\{z \in \mathbb{C} : |z| = 1\}$.

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