EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2021	MATH2031-WE01

Title:
Analysis in Many Variables II

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. To receive credit, your answers must show your working and explain your reasoning.

Revision:	

ED01/2021

Q1 **1.1** Let $\boldsymbol{x}$ be the position vector in $\mathbb{R}^3$, with $r = |\boldsymbol{x}|$, $\boldsymbol{a} \in \mathbb{R}^3$ a constant vector and $\boldsymbol{V}$ a smooth vector function of $\boldsymbol{x}$. Use index notation to simplify the following:

(i) $\boldsymbol{\nabla} \cdot (r\boldsymbol{a})$

(ii) $\boldsymbol{\nabla} \times (\boldsymbol{a} \times \boldsymbol{\nabla} r)$.

1.2 Define the term local diffeomorphism, and show that the map $\boldsymbol{v} : U \to \boldsymbol{v}(U) \subseteq \mathbb{R}^2$ for $\boldsymbol{v}(\boldsymbol{x}) = \left(xe^y, x^2 - y^3\right)$ is an orientation reversing local diffeomorphism for $U = \mathbb{R}^2 \setminus \{\boldsymbol{0}\}$.

1.3 In the following, all expressions are understood 'in the sense of distributions'.

(i) By integrating both sides against an arbitrary test function, find the coefficient A in the following identity

$$\delta(x^3 - \pi^3) = A\,\delta(x - \pi).$$

(ii) What is the general solution of the equation

$$g(x)\,(x^2 - 4)^2 = 0?$$

1.4 Let

$$L = \frac{d}{dx}\left(r(x)\frac{d}{dx}\right) + s(x)$$

be a differential operator with $r \in \mathcal{C}^1([a,b])$ and $s \in \mathcal{C}^0([a,b])$ for $[a,b]$ a bounded interval of $\mathbb{R}$. Assume that r and s are real-valued functions and that $r(x) \geq c$ for some $c > 0$. Consider the boundary value problem (BVP)

$$Lu(x) = f(x), \; a \leq x \leq b, \qquad B_1 u = B_2 u = 0,$$

where f is a source term, $u \in \mathcal{C}^2([a,b])$ and

$$B_1 u := u'(a) - \alpha u(a)$$
$$B_2 u := u'(b) + \beta u(b), \qquad \alpha, \beta > 0, \text{ real constants.}$$

Find the adjoint boundary conditions for this BVP and infer from your findings whether the BVP is self-adjoint or not. Justify any claim you make.

Q2 **2.1** Let $U \subseteq \mathbb{R}^n$ be an open set. State the conditions for a scalar field $f : U \to \mathbb{R}$ to be differentiable at a point $\boldsymbol{a} \in U$.

2.2 Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \left|x^2 - 1\right| \left(y^2 - 1\right).$$

(i) Where is f continuously differentiable?

(ii) Where is f differentiable?

2.3 Let $U = \{(x,y) \in \mathbb{R}^2 : x > 1\}$. With f as in question **2.2**, determine whether the curve $f : U \to \mathbb{R}$ given by $f(x,y) = c$ can be written in the form $y = g(x)$, and if not state the points (x_0, y_0) and corresponding values of c where problems occur.

Q3 Let $\boldsymbol{F}$ be the vector field on $\mathbb{R}^3$ given by

$$\boldsymbol{F}(\boldsymbol{x}) = \left(3y, 2x, \sqrt{z^2+1}\right),$$

and for $r \in \mathbb{R}$, P_r be the surface

$$P_r = \left\{(x,y,z) \in \mathbb{R}^3 : z = x^2 + \frac{y^2}{2} - r^2\right\}.$$

3.1 Calculate $\int_{S_r} (\boldsymbol{\nabla} \times \boldsymbol{F}) \cdot d\boldsymbol{A}$, where S_r is taken to have the downward pointing normal, and is:

(i) The part of P_r below the plane $z = 0$.

(ii) The part of P_r below the plane $z = 2x$.

3.2 Let

$$Q_r = \left\{(x,y,x) \in \mathbb{R}^3 : z^2 = 2(6x - x^2 + 3r^2) - 3y^2\right\}.$$

Calculate $\int_{S_r} (\boldsymbol{\nabla} \times \boldsymbol{F}) \cdot d\boldsymbol{A}$, where S_r is the part of the surface Q_r above the plane $z = 2x$, and Q_r is taken to have outward pointing normal. (Hint: Use Stokes' theorem.)

Comment on your result in relation to the divergence theorem.

3.3 Let $\boldsymbol{G}(\boldsymbol{x})$ be the vector field given by

$$\boldsymbol{G}(x,y,z) = \left(\frac{-y}{(x+z)^2+y^2}, \frac{x+z}{(x+z)^2+y^2}, \frac{-y}{(x+z)^2+y^2}\right),$$

and let γ be the circle in the $z = 0$ plane centred on the origin with radius $\frac{1}{2}$. Compute $\int_{\gamma_u} \boldsymbol{G} \cdot d\boldsymbol{x}$ and $\int_{\gamma_l} \boldsymbol{G} \cdot d\boldsymbol{x}$, where γ_u is the 'upper' semi-circle with $y \geq 0$, oriented from $(\frac{1}{2}, 0)$ to $(-\frac{1}{2}, 0)$, and γ_l is the 'lower' semi-circle with $y \leq 0$, oriented from $(-\frac{1}{2}, 0)$ to $(\frac{1}{2}, 0)$.

Comment on your result in relation to path independence of line integrals.

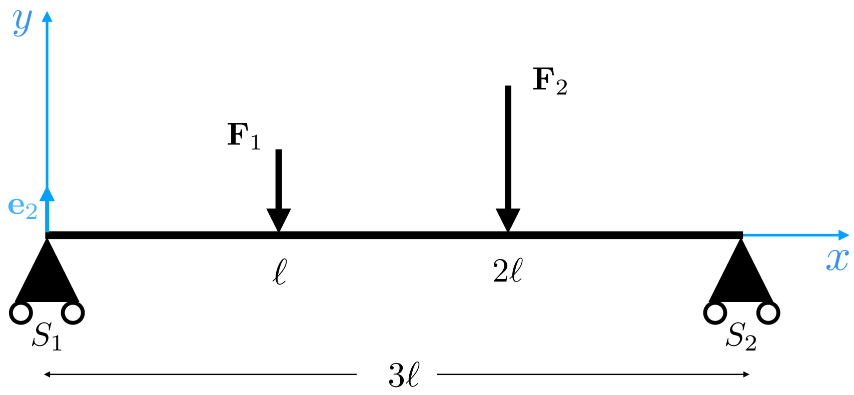
Q4 (i) A beam of length 3ℓ is resting on two simple supports S_1 and S_2 and is subjected to two point loading forces $\mathbf{F}_1 = -F_1\,\mathbf{e}_2$ and $\mathbf{F}_2 = -F_2\,\mathbf{e}_2$, ($F_1, F_2$ constants), applied at one-third and two-thirds of the length of the beam respectively (see Figure below). The equilibrium of the beam is expressed by the following relations, which involve two functions of x, called the bending moment $M(x)$ and the shear force $V(x)$, and which must be understood 'in the sense of distribution',

$$\frac{dV}{dx} - F_1\,\delta_\ell - F_2\,\delta_{2\ell} = 0,$$
$$\frac{dM}{dx} + V = 0.$$

Express V and M in terms of the Heaviside distribution Θ, defined as

$$\Theta(x) := \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x \leq 0. \end{cases}$$

and given the conditions $M(0) = M(3\ell) = 0$.



(ii) Consider the following partial differential equation 'in the sense of distributions',

$$\frac{\partial}{\partial t}u(x,t) + c\frac{\partial}{\partial x}u(x,t) + au(x,t) = \delta(x)\,\delta(t),$$

where x and t are one-dimensional real variables and a, c are real constants. Solve this equation by the Fourier transform method, i.e. (1) Fourier-transform the differential equation in the x variable; (2) seek the generalised function which solves the equation in Fourier space; (3) apply the inverse Fourier transform to the solution in Fourier space to obtain the solution to the initial differential equation in (x, t) space.

For $c = 1$ and $a = 0$, rewrite your solution $u(x, t)$ so that it is explicitly symmetric in x and t.

Hint: *in step 2, you might want to identify the Fourier space solution to the homogeneous equation first, and then determine which generalised function it should be multiplied by in order to obtain the actual solution of the equation in Fourier space.*

Q5 (i) (a) Use the divergence theorem to show the identity

$$\int_V (\,\phi\nabla^2 G - (\nabla^2\phi)G\,)\,dV = \int_S (\,\phi\boldsymbol{\nabla} G - (\boldsymbol{\nabla}\phi)G\,)\cdot d\mathbf{A}$$

where the surface S is the boundary of the volume V in $\mathbb{R}^3$.

(b) Let $\mathbf{x} \in \mathbb{R}^3$. Suppose that ϕ satisfies the partial differential equation

$$\nabla^2\phi(\mathbf{x}) = \rho(\mathbf{x})$$

with boundary conditions $\phi(\mathbf{x})\Big|_{\mathbf{x}\in S} = F(\mathbf{x})$, and that G satisfies the Green's function equations

$$\nabla^2 G(\mathbf{x},\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y}), \qquad G(\mathbf{x},\mathbf{y})\Big|_{\mathbf{x}\in S} = 0.$$

Use the above identity to obtain an expression for $\phi(\mathbf{x})$ in terms of ρ, F and G.

(c) Given that in $\mathbb{R}^3$,

$$\nabla^2 \frac{1}{||\mathbf{x}-\mathbf{y}||} = -4\pi\delta(\mathbf{x}-\mathbf{y}),$$

show that

$$G(\mathbf{x},\mathbf{y}) = -\frac{1}{4\pi\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}} + \frac{1}{4\pi\sqrt{(x_1+y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}}$$

satisfies

$$\nabla^2 G(\mathbf{x},\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y}) \quad \text{for } x_1 > 0,\ y_1 > 0, \quad G(\mathbf{x},\mathbf{y})\Big|_{x_1=0} = 0.$$

(ii) Let L be a differential operator of Sturm-Liouville type and consider the boundary value problem $Lu(x) = \lambda u(x) + f(x)$ for $a < x < b$ with a, b finite and real, and $u(a) = u(b) = 0$. The functions f and u are taken to be elements of $L^2([a,b])$, and the eigenvalues of L are given by $\lambda_1 < \lambda_2 < ... < \lambda_k < ...$, with real-valued orthonormal eigenfunctions $\{\varphi_1, \varphi_2, \ldots, \varphi_k, ...\}$.

Suppose that $\lambda = \lambda_k$ in the differential equation above. Show that, if $\langle f, \varphi_k\rangle = \int_a^b f(x)\varphi_k(x)dx = 0$, then a solution of that differential equation is given by

$$u_k(x) = \int_a^b G(x,y;\lambda_k)\, f(y)dy, \quad G(x,y;\lambda_k) = \sum_{\substack{n=1 \\ n\neq k}}^{\infty} \frac{\varphi_n(x)\varphi_n(y)}{\lambda_n - \lambda_k}.$$

Discuss whether this solution is unique or not.