Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2021	MATH2071-WE01

Title:

Mathematical Physics II

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. To receive credit, your answers must show your working and explain your reasoning.

Revision:	

ED01/2021

Q1 Consider a system described by the Lagrangian

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) + 2\cos(q_1 - q_2) - \sin(q_1 q_2)\,.$$

1.1 Write down the equations of motion for the system. (You do not need to solve them.)

1.2 Show that $q_1(t) = q_2(t) = 0$ is a solution of the equations of motion.

1.3 Assuming that the system starts at rest from the position $q_1(0) = q_2(0) = \epsilon$ for some $0 < \epsilon \ll 1$, find $q_1(t)$ and $q_2(t)$ to first order in ϵ.

Consider now the quantum mechanical simple harmonic oscillator of angular frequency ω and mass m. The annihilation operator is defined in terms of the position operator $\hat{x}$ and momentum operator $\hat{p}$ as

$$\hat{a} = \frac{1}{\sqrt{2m\hbar\omega}}\left(m\omega\hat{x} + i\hat{p}\right)\,.$$

1.4 Show that the wave function

$$\psi_0(x) = C\exp\left(-\frac{m\omega x^2}{2\hbar}\right)\,.$$

is the ground state wave function at a fixed moment in time. Also compute the constant C, and show how this wave function evolves in time, that is, find $\psi_0(x,t)$.

1.5 Prove that the wave function

$$\psi(x,t) = e^{-\lambda^2/2}\exp\left(\lambda\hat{a}^\dagger\right)\psi_0(x,t)\,,$$

for some real constant λ and $\phi_0(x,t)$ as above, satisfies the equation

$$\hat{a}\psi(x,t) = \lambda\psi(x,t)\,.$$

Q2 Consider a system described by a Hamiltonian of the form

$$H = \frac{f(x^2+y^2+z^2)}{2}(p_x^2 + p_y^2 + p_z^2) + V(ax^2 + y^2 + z^2, bt)$$

with Poisson brackets $\{x, p_x\} = \{y, p_y\} = \{z, p_z\} = 1$ and all other Poisson brackets vanishing. Here $a, b \in \mathbb{R}$ are two constants, and f and V are functions of one and two variables respectively.

2.1 Determine for which values of a, b the quantity $Q = xp_y - yp_x$ is conserved, assuming that V is generic.

2.2 Determine for which values of a, b the Hamiltonian H is conserved, assuming that V is generic.

2.3 Find the Lagrangian L_{ab} associated to the Hamiltonian above (for generic values of a and b).

2.4 Compute the infinitesimal transformation of L_{ab} generated by Q. Show that for the values of a, b you found in **2.1** Q generates a symmetry.

2.5 Coming back to the Hamiltonian formalism, choose

$$V(ax^2+y^2+z^2, bt) = \frac{1}{2}(x^2+y^2+z^2)$$

(note that in particular we set $a=1$) so that

$$H = \frac{f(x^2+y^2+z^2)}{2}(p_x^2+p_y^2+p_z^2) + \frac{1}{2}(x^2+y^2+z^2)\,.$$

Find another conserved charge P such that $\{P,H\}=0$ but $\{P,Q\}\neq 0$. Show, without computing any Poisson bracket explicitly, that $\{P,Q\}$ is conserved.

Q3 We have an infinite string made of two semi-infinite pieces joined at $x=0$. The two pieces have the same constant tension τ, but the piece at $x<0$ has constant density ρ_- while the piece at $x>0$ has constant density ρ_+. The corresponding Lagrangian density is thus

$$\mathcal{L} = \frac{1}{2}\rho_\pm (u_t)^2 - \frac{1}{2}\tau(u_x)^2$$

with the plus sign chosen for $x>0$ and the minus sign for $x<0$.

Consider the monochromatic wave ansatz given by

$$u(x,t) = \begin{cases} \Re\left((e^{ik_-x} + Re^{-ik_-x})e^{-ik_-c_-t}\right) & \text{for } x<0 \\ \Re\left(Te^{ik_+x}e^{-ik_+c_+t}\right) & \text{for } x>0 \end{cases}$$

where $\Re(z)$ denotes the real part of z and $k_\pm$ are real constants. We have also defined $c_\pm^2 = \tau/\rho_\pm$.

3.1 Provide, without doing any calculations specific to this example but explaining your reasoning, the values for R and T that you expect to obtain in the limits $c_+/c_- \to 1$ and $c_+/c_- \to 0$.

3.2 Solve for R and T in terms of c_+, c_-. Here you should take c_+ and c_- arbitrary, not just the limiting values considered in the previous part.

3.3 Show that the results you obtained in the previous part are compatible with the expected values you argued for in the first part, by taking the appropriate limits.

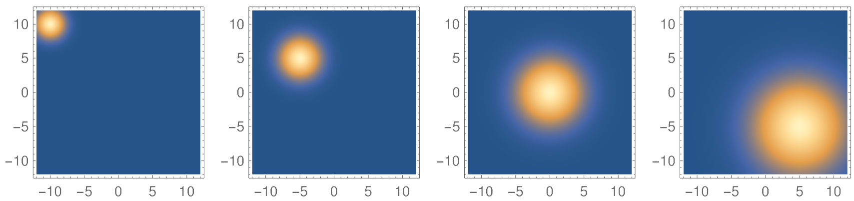
3.4 We now specialize to the case $c_-=c_+$, and set, at $t=0$, a initial configuration at rest of the form

$$u(x,0) = \begin{cases} e^{-2x/(x^2-1)} & \text{if } x\in[-1,1]\,, \\ 0 & \text{otherwise.} \end{cases}$$

Find $u(x,t)$ for all t. Is this a monochromatic wave? Justify your answer.

Q4 Consider two non-interacting quantum particles of mass $m = 1/2$, with coordinates x_1 and x_2 respectively, on the real line, $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$. Use units for which $\hbar = 1$.

4.1 The plots below are snapshots of the probability density $P(x_1, x_2, t)$ at times $t = 0, 1, 2, 3$ for the system described by a particular wave function $\psi(x_1, x_2, t)$, with x_1 along the horizontal axis, and lighter colours corresponding to larger values of $P(x_1, x_2, t)$. Which dynamical situation does this wave function describe? How would the plots change (qualitatively) if the particles were interacting?



Now restrict the system to a one-dimensional unit-size box, so that $0 < x_i < 1$ for both $i = 1, 2$.

4.2 Write down the expansion of a generic wave function for the two particles in the box, on a basis of normalised energy eigen-wavefunctions for this system.

4.3 Assume that at a particular time, the system is in the state described by the wave function

$$\psi(x_1, x_2, t = t_0) = C\left[\sin(2\pi x_1)\sin(2\pi x_2) + \frac{1}{2}\sin(3\pi x_1)\sin(\pi x_2)\right], \qquad (1)$$

for some normalisation constant C. Compute C.

4.4 Determine the probability density $P(x_1)$ for the system described by this wave function.

4.5 At $t = t_0$, the position of particle 2 is measured, and found to be $x_2 = 1/4$. Describe what you now know about the wave function, and give the probability density $P(x_1)$ just after this measurement.

4.6 What is the expectation value $\langle x_1 \rangle$ before the measurement, when the system is described by the wave function in equation (1)? Does the measurement of the position of particle 2 change this? Motivate your answer.

Q5 **5.1** Consider a unit-mass quantum mechanical particle in a box $0 < x < L$. We usually impose that the wave function vanishes at $x = 0$ and $x = L$, but there are more general boundary conditions such that the momentum operator is Hermitian. Show that

$$\langle \hat{p}\psi_1,\, \psi_2 \rangle = \langle \psi_1,\, \hat{p}\psi_2 \rangle$$

for the more general boundary condition that

$$\psi_i(x = 0) = e^{i\theta}\psi_i(x = L)\,,$$

where $\theta \in \mathbb{R}$.

5.2 Impose periodic boundary conditions $\psi(x=0) = \psi(x=L)$ (but typically non-zero). Show that $\chi_m(x) = C\exp(2\pi i m x/L)$ (with m an integer) are momentum eigenfunctions with eigenvalues p_m, and find C. Using these eigenfunctions, we can use $[\hat{x},\, \hat{p}] = i\hbar$ to 'prove' that $1 = 0$, as follows:

$$\begin{aligned} i\hbar = i\hbar\langle\chi_m, \chi_m\rangle &= \langle\chi_m, [\hat{x},\, \hat{p}]\chi_m\rangle = \langle\chi_m, \hat{x}\hat{p}\chi_m\rangle - \langle\chi_m, \hat{p}\hat{x}\chi_m\rangle \\ &= \langle\chi_m, \hat{x}\hat{p}\chi_m\rangle - \langle\hat{p}\chi_m, \hat{x}\chi_m\rangle = (p_m - p_m)\langle\chi_m, \hat{x}\chi_m\rangle = 0\,, \end{aligned}$$

where we used that $\hat{p}$ is Hermitian to go from the first to the second line. Make explicit what went wrong here.

5.3 Revert to 'standard' boundary conditions $\psi(x=0) = \psi(x=L) = 0$. Assume the system is described by the normalised wave function

$$\psi(x) = \sqrt{\frac{30}{L^5}}(L-x)x\,.$$

Decompose this function on the basis of energy eigenfunctions, that is, find the c_n in

$$\psi(x) = \sum_n c_n\phi_n(x)\,, \quad \text{with} \quad \phi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)\,.$$

You may use (and do not have to prove)

$$\begin{aligned} \int_0^L x\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x &= -\frac{L^2}{n\pi}(-1)^n\,, \\ \int_0^L x^2\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x &= -\frac{L^3}{n^3\pi^3}\Big(2 + (-1)^n(-2 + n^2\pi^2)\Big)\,. \end{aligned}$$

The function $\psi(x)$ is even around $x = L/2$; what does this mean for the coefficients c_n?

5.4 Compute $\langle E^2\rangle$ using the basis decomposition you just found, that is, compute

$$\langle E^2\rangle = \sum_{n=0} |c_n|^2(E_n)^2\,,$$

where E_n are the energy eigenvalues for the eigenfunctions $\phi_n(x)$. You may use (and do not have to prove)

$$\sum_{n\text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{8}\,.$$

5.5 You may have been tempted to compute the expectation value of E^2 by computing

$$\langle\psi, E^2\psi\rangle = \langle\psi, H^2\psi\rangle = \left(\frac{\hbar^2}{2m}\right)^2\int_0^L \overline{\psi(x)}\frac{\mathrm{d}^4}{\mathrm{d}x^4}\psi(x)\mathrm{d}x\,,$$

but then the result comes out as zero. Which of the two computations of $\langle E^2\rangle$ do you think is correct, and why?